Chapter 3: Situation Theory

Background

Jon Barwise and John Perry[1] initially presented situation theory (ST) in 1983 in *Situations and Attitudes* (S&A). Since then, Barwise and Perry have changed their thinking on several important issues presented in S&A. Also, there are now many other people working on their own versions of ST differing in significant ways from the material in S&A. S&A remains the most extensive presentation of ST and situation semantics (SS), and much of the material is still conceptually relevant even though formally obsolete.

There are several major publications about situation theory:

1979: Perry published "The Problem of the Essential Indexical", developing ideas that play a key role in the development of situation semantics. [Perry 1979]

1981: Dretske published *Knowledge and the Flow of Information*, a major influence on Barwise and Perry. [Dretske 1981]

1983: Barwise and Perry published *Situations and Attitudes*, the first major publication in situation theory and semantics. [Barwise&Perry 1983]

1987: Barwise and Etchemendy published *The Liar: An Essay on Truth and Circularity*. [Barwise&Etchemendy 1987]

1988: Peter Aczel published *Non-Well-Founded Sets*, the source of Barwise's revised metatheory. [Aczel 1988]

1988: Barwise published *The Situation in Logic*, a collection of many of his papers spanning from 1981 to 1988. [Barwise 1988]

1990: Robin Cooper, Kuniaki Mukai, and John Perry then edited *Situation Theory and its Applications*, a collection of papers which represent work that evolved out of the First Conference on Situation Theory and Its Applications, March, 1989. [Cooper, et. al. 1990]

1991: Keith Devlin published *Logic and Information*. [Devlin 1991]

[1] [Barwise&Perry 1983]

The particular formulation for situation theory that is used in this work is based on material in *Logic and Information* [Devlin 1991], *Situation Theory and Its Applications* [Cooper, et. al. 1990], *The Situation in Logic* [Barwise 1988], and *Situations and Attitudes* [Barwise&Perry 1983]. Some material from [Barwise&Perry 1983] is updated to use the more modern formalisms found in [Cooper, et. al. 1990]. The most concise and readily understood presentation of issues underlying situation theory can be found in Barwise's "Situations, Facts, and True Propositions"[2]. In this paper, Barwise presents several points about what situation theory should be. The new expression of situation theory presented below addresses several of these points.

[2] p. 221 - 254 in [Barwise 1988].

Informal Overview

Situation theory is addressed to problems of information and meaning - how can people mean, how can they possess and communicate information, what is meaning, what is information. It also addresses mental states such as knowledge and belief, and mental processes of inference and perception.

Situation theory is based on the idea that all of the issues of information and meaning must be understood in the light of the "reasoning agent" being *situated* in the world, and that the most basic concept in the analysis is the "situation" - some part of the world (generally, a *part* of the world "accessible" to the agent). By definition, one can determine the "state of affairs" with respect to a situation; either a state of affairs holds in some situation or it does not. If a state of affairs holds with respect to some part of the world, a situation, then that situation is said to *support* that state of affairs. A state of affairs is *actual* if it is supported by at least one situation. An *actual* state of affairs can be called a *fact*. The neutral technical term in situation theory for states of affairs is *infon.* This term is adopted to refer to the idea that infons are the basic units of information.

The major result of this chapter is the development of a comprehensive definition of the supports relation, and an axiom system for infons which characterizes the supports relation. This chapter proves the first hypothesis set forth in chapter 1:

> *First Hypothesis*: A version of situation theory can be defined which has a characterizing logic (an "infon" logic) similar in form and expressivity to classical first order logic.

The definition of the *supports* relation is central to the definition of situation theory. The *supports* relation presented here respects four postulates:

1. *Coherence postulate*: No real situation s supports a state of affairs and its negation.
2. *Compatibility postulate*: For any two (real) situations s_1 and s_2, there is a (real) situation s such that s_1 and s_2 are portions of s.

3. *Persistence postulate*: If s_1 is a portion of s then any state of affairs [infon] that holds in s_1 also holds in s.
4. *Duality postulate*: For every state of affairs [infon], there is a state of affairs [infon] that is its dual, or negation.

Barwise identifies the first three as “cherished principles” of situation theory.[3] He was willing to forego the third postulate, persistence, in order to introduce the “thesis” that every infon (state of affairs) has a dual (negation). He felt it necessary to violate “persistence” to introduce the dual of an existentially quantified infon (i.e. a universally quantified infon). This thesis takes a different approach to infon logic and situation theory which preserves all four postulates. Particularly noteworthy in this regard is that the persistence property holds for *all* infons, including existentially and universally quantified infons. Also, *every* infon has a dual. It is unique to the situation theory developed in this thesis that all of these postulates hold.

Formal Presentation

The version of situation theory developed here is a full analog of classical first order predicate logic, with semantics for $\wedge$, $\vee$, $\Rightarrow$, $-$, $\exists$, and $\forall$ (conjunction, disjunction, conditional, negation, existential, and universal operators, respectively). This is not all of situation theory. Elements of situation theory not addressed by this thesis include: abstract relations, partial infons, parameters as primitive objects, restriction (of parameters), and higher order infon logic. These elements of situation theory are all areas for further development of the research on which this thesis reports.

There are two major elements of situation theory, a definition of when a situation “supports” an infon and an axiom system for infons. The axiom system for infons is determined by the supports relation: If a situation supports an infon, and one can derive via the axiom system a second infon from the first infon, then the situation must also support the second infon. Thus, the axiom system must be defined in such a way that it preserves the “supports” relation. This is the “soundness” requirement for the axiom system. The other major requirement for the axiom system is that any infon

[3] p. 235 in [Barwise 1988].

that is supported by all situations be derivable as a theorem of the axiom system. This is the "completeness" requirement for the axiom system. This chapter proves that the axiom system is sound. The non-quantificational infon logic axiom system is shown to be complete with respect to the non-quantificational conditions for the *supports* relation. It is conjectured that the quantificational axiom system is also complete.

The "axioms" in the infon axiom system are compound infons that are supported by *all* situations. They are tautologies with respect to the supports relation.

The following discussion develops definitions for the supports relation and an axiom system, and argues that the defined supports relation is consistent with the ideas of situation theory and that the axiom system is support-preserving. This is a complex presentation and is done in several parts. Several concepts are defined in addition to the supports relation, the language of infon logic, and the axiom system for infon logic. These additional concepts include:

axiom system (also known as "Hilbert system"),
Scott consequence relation ("SCR"),
Tarski consequence relation ("TCR"),
propositional Kripke structure,
strong propositional Kripke structure,
Kripke structure,
and strong Kripke structure.

Also, several axiom systems are presented:

H – Heyting's Predicate Calculus,
h – the propositional fragment of H,
h$^-$ – the negation-free propositional fragment of H,
H^{++} – the conditional-only fragment of H,
nh – the strong negation propositional fragment of H, and
NH – the strong negation version of H.

All of the axiom systems identified above, except NH, are presented by Gabbay[4]. NH is developed in this thesis.

The steps of the presentation of the development of the supports relation and the infon logic are summarized below:

1) The supports relation and the infon logic axiom system definitions are previewed. The subsequent discussion develops these definitions and argues that the defined infon axiom system *characterizes* the defined supports relation.
2) The language of infon logic is defined.
3) The conditions on the supports relation are given that relate to *propositional connective-free* infons.
4) The *strong propositional Kripke structure* for semantic interpretation is introduced.
5) The *supports* relation definition is extended with conditions defining the support of confirmation and denial of conjunction and disjunction.
6) The *conditional-free* supports relation is shown to define a conditional-free strong propositional Kripke structure.
7) The *Scott* and *Tarski consequence relations* are defined.
8) The supports relation conditions for implication are developed with regard to a "minimal" concept of implication with respect to the Scott consequence relation.
9) *Strong negation* axioms are introduced into the H system to give the NH system.
10) The propositional supports relation is shown to define a strong propositional Kripke structure.
11) The propositional fragment with strong negation of Heyting's predicate calculus is shown to be the "supports-preserving" axiom system for the propositional supports relation.
12) The *quantificational* conditions for the supports relation are given.
13) *Heyting's Predicate Calculus,* axiom system H, is presented.
14) The *quantified Kripke structure* is introduced.
15) The *strong Kripke structure* is defined.

[4] [Gabbay 1981]

16) The full supports relation is shown to define a strong Kripke structure.
17) The NH axiom system is proposed as the supports-preserving axiom system for the full supports relation.

1) The supports relation and infon logic axiom system are previewed

The entire supports relation and the infon logic axiom system are given below. These are developed and explained in detail in the following discussion. The supports relation is presented in Exhibit 3. 1 on page 27. To briefly present the notation and definitions used in Exhibit 3. 1: *s* and *t* are used to stand for situations, and σ and τ stand for infons. The dual of an infon σ is written $\overline{\sigma}$. As is discussed below, it is part of the design of the infon logic that the negation of an infon is logically equivalent to the dual of that infon. The symbol '|=' is read "supports". The symbol $\leq_S$' is read "is part of" and relates two situations. A *parameter* is an infon logic variable. It can have at most one value "mapped" to it. An *anchoring* is a function from parameters to infon logic terms which specifies a set of such bindings. A *non-parametric* anchoring is one for which the range of the anchoring function does not contain any parameters. This is the only kind of anchoring considered in this thesis. A more extended version of the situation theory presented here would incorporate parameters as "first class" terms in the logic, allowing one to quantify over them and establish relations between them, etc. Here they only serve their classically limited purposes. The *constituents* of a situation are all of those "things" which appear as arguments for any of the infons which that situation supports.

The infon axiom system NH is presented in Exhibit 3. 2 on page 28. The variables *A*, *B*, and *C* stand for any infon logic formula. These infon formula schemes are *axioms* in the sense that for any instance of one of these formula schemes, that instance is supported by all situations. An *instance* of an infon formula scheme is created by mapping the formula variables of the scheme to well-formed infon logic formulae, one formula per distinctly named variable of the scheme. The schematic formulae $A(x)$ and $A(y)$ stand for any well-formed infon logic formula which has *x* or *y* as a free parameter, respectively. A parameter is *free* if it is not in the scope of an existential or universal operator which binds it.

Support postulate 1: *Confirmation of Basic Infons*: For all situations s and basic infons σ, $s \models \sigma$ iff the basic infon σ is a state-of-affairs which obtains in the situation s. Equivalently, $s \models \sigma$ iff the situation s carries the information σ. (see page 36)

Support postulate 2: *Denial of Basic Infons*: For all situations s and basic infons σ, $s \models \overline{\sigma}$ iff the basic infon σ is a state-of-affairs such that the dual of σ obtains in the situation s. (see page 36)

Support condition 1. *Consistency of Basic Infons*: For all situations s and basic infons σ, $s \not\models \sigma$ *or* $s \not\models \overline{\sigma}$. This can also be stated as: it is *not* the case that $s \models \sigma$ *and* $s \models \overline{\sigma}$. (see page 36)

Support condition 2. *Partiality of Basic Infons*: For all situations s such that s is not the entire world, there exists a basic infon σ such that $s \not\models \sigma$ *and* $s \not\models \overline{\sigma}$. (see page 36)

Support condition 3. *Persistence of Basic Infons*: For all situations s, s', and basic infons σ, (if $s \leq_S s'$ and $s \models \sigma$, then $s' \models \sigma$). (see page 37)

Support condition 6. *Confirmation of Conjunction*: $s \models \sigma \wedge \tau$ iff $s \models \sigma$ and $s \models \tau$. (see page 41)

Support condition 7. *Denial of Conjunction*: $s \models -(\sigma \wedge \tau)$ iff $s \models \overline{\sigma}$ or $s \models \overline{\tau}$. (see page 42)

Support condition 8. *Confirmation of Disjunction*: $s \models \sigma \vee \tau$ iff $s \models \sigma$ or $s \models \tau$. (see page 42)

Support condition 9. *Denial of Disjunction*: $s \models -(\sigma \vee \tau)$ iff $s \models \overline{\sigma}$ and $s \models \overline{\tau}$. (see page 42)

Support condition 10. *Confirmation of Conditional*: $s \models \sigma \Rightarrow \tau$ iff for all t such that $s \leq_S t$, $t \models \sigma$ implies $t \models \tau$. (see page 48)

Support condition 11. *Denial of Conditional*: $s \models -(\sigma \Rightarrow \tau)$ iff $s \models \sigma$ and $s \models \overline{\tau}$. (see page 48)

Support condition 12. *Confirmation of Universal Quantification*: $s \models \forall \mathbf{x}\sigma$ iff for all situations t and non-parametric *anchorings* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t), $s \leq_S t$ implies $t \models \sigma[\boldsymbol{f}_t]$. (see page 50)

Support condition 13. *Denial of Universal Quantification*: $s \models -\forall \mathbf{x}\sigma$ iff there exists a non-parametric *anchoring* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in$ *constituents*(s), such that $s \models -\sigma[\boldsymbol{f}]$. (see page 50)

Support condition 14. *Confirmation of Existential Quantification*: $s \models \exists \mathbf{x}\sigma$ iff there exists some non-parametric *anchoring* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in$ *constituents*(s) such that $s \models \sigma[\boldsymbol{f}]$. (see page 50)

Support condition 15. *Denial of Existential Quantification*: $s \models -\exists \mathbf{x}\sigma$ iff for all situations t and non-parametric *anchorings* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t), $s \leq_S t$ implies $t \models -\sigma[\boldsymbol{f}]$. (see page 50)

Exhibit 3. 1: Definition of the Supports Relation.

$NH_0 = \{$

$A \Rightarrow (B \Rightarrow A),$
$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)),$
$A \wedge B \Rightarrow A,$
$A \wedge B \Rightarrow B,$
$A \Rightarrow (B \Rightarrow A \wedge B),$
$A \Rightarrow (A \vee B),$
$B \Rightarrow (A \vee B),$
$(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)),$
$A(x) \Rightarrow \exists y\, A(y),$
$\forall y\, A(y) \Rightarrow A(x),$
$--A \Leftrightarrow A,$
$-(A \Rightarrow B) \Leftrightarrow A \wedge -B,$
$-(A \wedge B) \Leftrightarrow -A \vee -B,$
$-(A \vee B) \Leftrightarrow -A \wedge -B,$
$A \wedge -A \Rightarrow B,$
$-\exists x A(x) \Leftrightarrow \forall x\, -A(x),$
$-\forall x A(x) \Leftrightarrow \exists x\, -A(x)\}$

For x not free in B:
$NH_1 = \{$

(A, A⇒B/B),
$(A(x) \Rightarrow B \,/\, \exists x A(x) \Rightarrow B),$
$(B \Rightarrow A(x) \,/\, B \Rightarrow \forall x A(x))\}$

$NH_2 = \{(\{A,\ A \Rightarrow B\}/\{B\})\}$

$A \Leftrightarrow B$ is defined to be a notation for $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Exhibit 3. 2: Hilbert Axiom System for Infon Logic.

2) The language of infon logic is defined.

The primitive terms in ST are *situations*, *relations*, and *objects*. The full ST also includes *parameters* as primitive objects. There are several special primitive relations, *supports*, *involves*, *precludes*, *material implication*, and a number of location relations (*precedes*, *temporally overlaps*, *spatially overlaps*, *temporally contains*, *spatially contains*, *temporally overlappingly precedes*). The composite terms of ST are *infons* , *object types*, *situation types*, and *propositions*. A *situation* is a part of the real world. A *relation* is a property of an n-tuple (n ≥ 1) of terms. *Objects* are “individuated" parts of the world.[5] A departure of ST from traditional logic is that relations are primitive, they are *intensional* concepts, not defined by their extension. Thus, there can be two different relations that have different intensions and the same extension. This is not possible in the traditional logical formulation, if two relations have the same extension then they must be the same relation.

There are two kinds of formulae in ST, propositions and *infons*. A proposition is either true or false and corresponds roughly with traditional logic meta-logic statements. The logic of propositions is that of first order predicate calculus. An infon is neither true nor false in isolation, but is supported or unsupported (true or false) with respect to a situation. An *infon* is *basic* or *compound*. An *infon* is composed of a *relation*, a set of (role-named) arguments, and a *polarity*. If the relation of the infon is *any* relation except the infon logic connectives ($\wedge_I$, $\vee_I$, $-$, $\forall_I$, $\exists_I$, and $\Rightarrow_I$), then the infon is a *basic infon*. As an example, to make the basic infon that block A is on block B, which might be written '$On(A, B)$' in a first order theory, one might write '$\langle\langle On$, [location:L, top:A, bottom:B]; $+\rangle\rangle$'. The "+" in the example basic infon is the *polarity* of the infon. Polarities are either positive or negative (+ or -). An infon with positive polarity is used to claim that the relation holds with respect to the argument values, the negative polarity is used to claim that the relation does *not* hold with respect to the argument values. The role-names ("top" and "bottom" in the example) allow one to specify a *partial infon*, one that does not have argument values for all of the roles defined for the infon's relation. Thus, an infon that simply said that A was on something could be written as $\langle\langle On$, [location:L, top:A]; $+\rangle\rangle$. There is not a consensus in the situation theory literature on the meaning of partial infons, and there is some disagreement as to whether the concept is needed. They do not appear to be necessary for the investigations of this thesis in belief and perception. Thus, this issue of partial infons is not explored in this thesis. That is, all of the infons are complete. Thus, basic infons are written using positional arguments instead of named arguments. So, the example basic infon becomes: $\langle\langle On, L, A, B; +\rangle\rangle$.

[5] Individuation is only discussed in three places in the ST literature, and then briefly.One reference is in an appendix of "Situations, Facts, and True Propositions" on pp. 251-253 of [Barwise 1988]. Another is in "Notes on Branch Points in Situation Theory" on pp. 260-261 of [Barwise 1988]. Finally, there is a discussion in [Devlin 1991].

Most relations have an explicit *location* argument, which specifies the time and space about which the infon "speaks". By convention, this argument will be the first argument. For some relations the "time" aspect of the location is all that is relevant, and for others the "space" aspect is all that is relevant. Because of this some presentations of ST handle time and space as independent quantities and independent arguments. It is convenient in this thesis to leave them combined.

Compound Infons

Basic infons can be used to make *compound infons*. The "logic" of these compound infons is called *infon logic.* Infon logic has several connectives and quantifiers: $\wedge_I$, $\vee_I$, $-$, $\forall_I$, $\exists_I$, and $\Rightarrow_I$ (conjunction, disjunction, negation, universal quantification, existential quantification, and conditional, respectively). The "I" subscript of the symbol identifies the symbol as being used in infons. The meanings of these connectives must be analyzed in terms of the 'supports' relation - analyzing under what circumstances a situation supports a compound infon using these symbols. This analysis yields some different results from the meaning associated with these symbols in classical logic, closer to the meanings used in intuitionistic logic.

The *dual* of an infon is another infon with the same relation and arguments, but the other polarity. An infon is equivalent to the dual of its dual.[6]

A compound infon is an infon that has one of the infon logic connectives as its relation, and has arguments appropriate to that connective.

An inductive definition of well-formed infons is:
Basis: If A is a basic infon, then it is a well-formed infon.
Let A and B be well-formed infons, ***x*** be a parameter, and *i* be either polarity ('+' or '-');
1) $\langle\langle -, A; i \rangle\rangle$ is a well-formed infon
2) $\langle\langle \wedge_I, A, B; i \rangle\rangle$ is a well-formed infon

[6] Different developments of ST are possible where an infon and the dual of its dual are not (necessarily) equivalent. This is explained in some detail in [Barwise&Etchemendy 1990].

3) $\langle\langle \vee_I$, A, B;i $\rangle\rangle$ is a well-formed infon

4) $\langle\langle \Rightarrow_I$, A, B;$i\rangle\rangle$ is a well-formed infon

5) $\langle\langle \exists_I$, $\boldsymbol{x}$, A;$i\rangle\rangle$ is a well-formed infon

6) $\langle\langle \forall_I$, $\boldsymbol{x}$, A;$i\rangle\rangle$ is a well-formed infon.

The notation defined for infon logic is translated into infons as follows:

1) '–A' is $\langle\langle$–, A;+$\rangle\rangle$.[7]

2) 'A $\wedge_I$ B' is $\langle\langle \wedge_I$, A, B; + $\rangle\rangle$.

3) 'A $\vee_I$ B' is $\langle\langle \vee_I$, A, B; + $\rangle\rangle$.

4) 'A $\Rightarrow_I$ B' is $\langle\langle \Rightarrow_I$, A, B; + $\rangle\rangle$.

5) '$\exists_I \boldsymbol{x}$A' is $\langle\langle \exists_I$, $\boldsymbol{x}$, A;+$\rangle\rangle$.

6) '$\forall_I \boldsymbol{x}$A' is $\langle\langle \forall_I$, $\boldsymbol{x}$, A;+$\rangle\rangle$.

In the discussion that follows, the "I" subscript is left off of the infon connectives to increase readability. It should be clear from context if a particular formula is meant to be read as an infon, in which case the connectives in the formula should be read as infon connectives. A well-formed formula in the infon logic is one which represents a well-formed infon, as presented above. This is given the common abbreviation of "wff" in the following discussion. The semantics of these compound infons is defined in the various Support Conditions, as summarized in Exhibit 3. 1 on page 27.

Situations

A *situation* is any space and time region of reality (the universe). Space and time are pre-theoretic concepts in this thesis - the common notions of these concepts are intended.

There is one relation defined on situations, "part of", written '$s \leq_S t$' and read "s is

[7] $-A$ is logically equivalent to the dual of A, $\overline{A}$. This is expressed by Support condition 4 on page 41 and Support condition 5 on page 41.

part of *t*". A situation *s* may be part of another situation *s'* in that the region identified by *s* may be contained in another ("larger") region *s'*. Some implications of this idea of "part of" are found in the following discussion of the *persistence* condition of the "supports" relation. A situation can be "modeled" by a *set* of infons - all of those infons that the situation supports in some scheme of individuation. Such a set of infons is called an *abstract situation.*

There is a temptation to represent the abstract situation via a conjunction in infon logic of the basic infons in this model set. There are two aspects of this set of infons that make this impossible, however. One is that a wff of infon logic must be finite, and the set of infons is infinite. The set of infons may have a finite set of axiom infons from which the entire set can be derived (via infon logic). This finite axiomatic infon set can be converted to a wff that is the conjunction of its contents, yielding a wff that implies the modeling set of infons. There need not be such a finite axiomatization of the model set, however - any situation, by virtue of being part of the real world, supports infinitely many independent facts about what exists at the infinitely many distinct locations in the space-time region of the situation if one's scheme of individuation allows for real/continuous locations (versus discrete locations).

A second problem with using a conjunction of basic infons to represent a situation stems from circular references. Since a situation can support infons that refer to the containing situation, the set of infons of a situation can be a circular set. Such a set is an example of a non-wellfounded set. This circularity cannot be expressed in a conjunction of basic infons, one needs an "infon parameter" that is equivalent to the entire conjunction and can be used as a term in the basic infons of the conjunction. Given this, one can define a recursive infon - but there is no mechanism in infon logic with which to reason about such an infon.[8] As an example, suppose *s* is a situation where Pat is looking in a mirror. Pat sees *s* (Pat sees Pat looking in the mirror). *s* supports the infon $\langle\langle$sees, Pat, *s*; +$\rangle\rangle$. In the situation theory of this thesis, this is a well-formed, simply handled infon since *s* is a reference to a situation. If instead one represents situations as conjunctions of basic infons, then *s* is an infon formula variable which requires some kind of higher-order logic to handle.

[8] A recursive wff is different from the reflective wffs of Z modal logic. There, all of the "recursive" references must be in modal expressions.

Other Elements of Infon Logic

The idea of the partial infon is approximated by existential infons. The example partial infon can be represented as $\langle\langle\exists, x, \langle\langle On, A, x; +\rangle\rangle; +\rangle\rangle$ (As an infon formula: '$\exists x\ On(A, x)$'). This is subtly different from the partial infon in that the partial infon can be supported by a situation that "says nothing" about the unspecified argument roles, but an existential infon is not supported by such a situation. An existential infon which has a basic infon as its infon formula argument requires that *something* be said about *all* of the arguments to that basic infon. This gives different approaches to the knowledge representation problem of indeterminacy vs. semantic primitives identified by Barr&Feigenbaum and the incompleteness item of the Brachman&Levesque basic epistemology category.

Having the *situation* as part of the theory is an important departure from traditional logics. The analogous concept for a combination in ST of a situation and an anchor in traditional logic is the "model" or "interpretation", which is a meta-logical concept.[9] In the initial presentation of ST, situations (and anchors) were *not* part of the logic, instead, *abstract* situations were present in the logic. These abstract situations are finite, partial, characterizations of real situations via "sentences" of that theory.

The *constituents* of a situation are all of the objects that are values of arguments for any of the infons supported by that situation. One of the important points of ST is that this collection of constituents can be self-referential - the situation may itself be referred to as the value of an argument of an infon. Thus, the situation may be one of its own constituents. This is one reason for taking situations as primitives in ST. The extension of a situation is the collection of all of the infons that the situation supports. An infon's argument can be a reference to the entire situation which supports that infon. The set of constituents of such a situation contains the situation. This circularity can be described by a kind of set called a *non-wellfounded* set[10].

[9] The Z modal logic of Frank Brown is also an exception in this area in that the "world" (as in a "possible world" semantics of a traditional modal logic) has an explicit representation in the logic, instead of being meta-logical.

[10] This set theory is presented in [Aczel 1988].

A *situation type*, written $[s \mid S\ ]$ where S is an infon, defines a collection of all situations s such that $s \models S$.

An *object type* or *abstract relation*, written $[x \mid \mathrm{P}(x)]$ where P is an infon free in argument-value-uses of parameter x, defines a collection of all objects in a situation s where $s \models$ P. This can be viewed as a lambda-like expression where:

$s \models \langle\langle [x \mid \mathrm{P}(x)], \mathrm{a};+\rangle\rangle$ iff $s \models \mathrm{P(a)}$,

where $\mathrm{P}(x)$ is some infon free in x , 'a' is some constant, and P(a) is P with all free occurrences of x replaced by 'a'.[11] An example of a use of this notation is given by Barwise gives an example where the infon $\langle\langle$admires, a, b;+$\rangle\rangle$ is used to form two derivative properties, 'admiring b' and 'being admired by a' :

(admires b) = $[x \mid \langle\langle \text{admires}, x, \text{b}; +\rangle\rangle]$

(admired by a) = $[x \mid \langle\langle \text{admires}, \text{a}, x; +\rangle\rangle]$[12]

Barwise makes the point that this gives rise to three distinct (syntactically) infons which are strongly equivalent - if any one of them is supported by a situation, then all three of them are supported by that situation. These three infons can be written as:

$\langle\langle$admires, a, b;+$\rangle\rangle$

$\langle\langle$ (admires b), a;+$\rangle\rangle$

$\langle\langle$ (admired by a), b;+$\rangle\rangle$

Abstract relations are not dealt with formally in the following material.

The *parameter* as an explicit element of the full ST language is another important deviation from traditional logics, where parameters (or variables) are part of the meta-language. This allows one to reason about parameters, and their presence or absence, within ST. This is part of making ST a fully reflective logic. The parameter can be *restricted* to a particular domain by applying a restricting infon to it that also contains that parameter. Only values of the parameter that makes the restricting infon hold are in the (restricted) domain of the parameter.

Since parameters are part of the language of ST, the associating of parameters with

[11] This notation and interpretation is presented on p. 233 of [Barwise 1988].

[12] p. 233 in [Barwise 1988].

values must also be represented. This associating "function" is called an *anchor* (it anchors the parameters). An anchor is a function that takes an infon as an argument and returns that infon with the appropriate parameter substitutions.

The *involves* relation, written $[s \mid S] \Rightarrow_{\text{inv}} [t \mid T]$, is a proposition that asserts that whenever there exists a situation of type $[s \mid S]$, then there also exists a situation of type $[t \mid T]$. There is a strong relationship between the *involves* relation '$\Rightarrow_{\text{inv}}$' and the infon conditional '$\Rightarrow_{\text{I}}$': For all bindings of the parameters in S and T, $[s \mid S] \Rightarrow_{\text{inv}} [t \mid T]$ is true iff $S \Rightarrow_{\text{I}} T$ is factual (i.e. is supported by *some* situation). The *precludes* relation, written $[s \mid S] \perp [t \mid T]$, is a proposition that asserts that whenever there exists a situation of type $[s \mid S]$, then there does not exist a situation of type $[t \mid T]$. This is the opposite of the *involves* relation. This is related to the infon conditional by: For all bindings of the parameters in S and T, $[s \mid S] \perp [t \mid T]$ is true iff $S \Rightarrow_{\text{I}} T$ is not factual (i.e. is not supported by *any* situation).

There is a conditional form of the involves and precludes relations, $\text{S} \Rightarrow_{\text{inv}} \text{T} \mid \text{R}$ and $\text{S} \perp \text{T} \mid \text{R}$ respectively, where the relation does not hold unless the condition situation type (R) holds. Different situation types in the same constraint may share parameters. Thus, a binding of some shared parameter in R specializes S and T. This introduces the idea of parametrized infons and situation types.

Having presented these various aspects of infon logic, the propositional elements are focussed on in the following material, then quantification is added. The infon logic axiom system does not deal with *involves*, *precludes*, *parameters*, and *situation types*. Parameters are only dealt with in the traditional metalogical fashion. This thesis does not address parameter restriction.

3) The conditions on the supports relation are given that relate to propositional connective-free infons.

The *supports* relation, written $\models$, is between situations and infons: $s \models \sigma$, which is read "s supports σ", s a situation and σ an infon. The negation of the relation, written $\not\models$, is an abbreviation of "it is not the case that $s \models \sigma$": $s \not\models \sigma$ is read "s does not support σ". The supports relation for situations s and *basic* (and non-parametric) infons σ is characterized by the following "intuitions" and "conditions":

Support postulate 1. *Confirmation of basic infons*: For all s and σ, $s \models \sigma$ iff the basic infon σ is a state-of-affairs that obtains in the situation s. Equivalently, $s \models \sigma$ iff the situation s carries the information σ.[13]

Support postulate 2. *Denial of basic infons*: For all s and σ, $s \models \overline{\sigma}$ iff the basic infon σ is a state-of-affairs such that the dual of σ obtains in the situation s.[14] (As mentioned above, the dual of σ is the same infon as σ but with the opposite polarity.)

Support condition 1. *Consistency of basic infons*: For all s and basic infons σ, $s \not\models \sigma$ *or* $s \not\models \overline{\sigma}$.[15] This condition on the supports relation requires that situations are *consistent* - a situation can't confirm and deny the same piece of information. This condition derives from situations being part of the *real* world, and in non-quantum analyses of the real world a state of affairs can't be both "the case" and "not the case" in a single situation. This is another statement of the *Coherence postulate*.

Support condition 2. *Partiality of situations*: For all s such that s is not the entire world, there exists a basic infon σ such that $s \not\models \sigma$ *and* $s \not\models \overline{\sigma}$.[16] This condition on the supports relation is a direct consequence of the *partial* nature of situations. Since a situation is *part* of the real world, there are facts about the real world about which the situation has nothing to say - that the situation neither confirms nor denies.

[13] This notion of "carrying information" is presented in [Barwise&Perry 1983].

[14] A discussion of the support of the dual of an infon can be found on p. 234 of [Barwise 1988]. Here Barwise argues for every infon having a dual, as is done in the situation theory of this thesis.

[15] The consistency of situations is discussed on p. 235 of [Barwise 1988].

[16] The partiality of situations is discussed in many places, since this is a statement of one of the fundamental tenets of situation theory. For instance, a discussion can be found on p. 234 of [Barwise 1988].

Support condition 3. *Persistence of basic infons*: For all s, s' , and basic infons σ, (if $s \leq_S s'$ and $s \models \sigma$, then $s' \models \sigma$).[17] This condition requires that if a situation carries a piece of information, then any "larger" situation of which it is part also carries that piece of information. One can say that an infon *persists* from a situation s that supports it into any situation of which s is a part. This is a restatement of the *Persistence postulate*.

If the situation under consideration is the entire universe (or *world*) w, then Support postulate 1, Support postulate 2, and Support condition 1 are all true of that situation. The negation of Support condition 2, totality instead of partiality, applies to w. Support condition 3, persistence, simply doesn't apply since w is defined to not be part of any situation but itself.

Define a *truth valuation function t* over infons by: $t_w(\sigma) = 1$ if $w \models \sigma$, and 0 if $w \models \overline{\sigma}$. Since w is total (negation of Support condition 2) and consistent, t_w is a total function over basic infons. Thus, t_w is a *model* in classical propositional logical terms if the basic non-parametric infons are considered as atomic propositions. Support postulate 1, Support postulate 2, and Support condition 1, and negation of Support condition 2 are what *should* hold of a truth valuation (with some appropriate rephrasing). This is not to say that there is an equivalence between all of classical propositional logic and situation theory. At this point in the discussion, no connectives have been defined for basic infons so no conclusions can be drawn about the similarity to classical connectives. But, there is this simple similarity for the fragment of classical propositional logic that has only atomic propositions and negation and the fragment of situation theory that has only non-parametric basic infons, duals, and the one situation w, the whole universe.

The truth valuation function defined over an arbitrary situation s that is not w has properties that are similar to those of a model for *intuitionistic* proposition logic. In this case, Support postulate 1, Support postulate 2, Support condition 1, and Support condition 2 (instead of "not Support condition 2") all hold, appropriately re-

[17] A discussion of persistence can be found in pp. 235-236 in [Barwise 1988].

phrased. The status of the persistence condition, Support condition 3, with respect to intuitionism is not clear in this simple interpretation. To make a comparison some idea of there being a containment-based pre-ordering of models for intuitionistic logic is needed. This can be found in the Kripke interpretation discussed below.

4) The strong propositional Kripke structure for semantic interpretation is introduced.

The following discussion formalizes the relationship between the interpretation of the non-parametric basic infon fragment of situation theory and interpretations of intuitionistic and classical propositional logics. There are several general formal mechanisms for discussing the interpretation of logic. Three of these mechanisms and their relationships to various logics, particularly the intuitionistic and classical propositional and predicate logics, are presented in detail in [Gabbay 1981]. Gabbay presents the Kripke, Beth and Topological interpretations. He shows substantial equivalences between these different interpretations. This allows one to use any one of them, without loss of generality. Gabbay focuses on the Kripke interpretation.

A *propositional Kripke structure* has the form (S, R, O, D)[18], where (S, R, O) is a pre-ordered set with a first element O and D is a function such that for each $t \in$ S and atomic q, D(t, q) $\in \{0, 1\}$. (R is the relation that pre-orders the elements of S. By the definition of a pre-ordering relation, R is a transitive and reflexive relation.) (S, R, O, D) has the persistence property: If tRs and D(t, q) = 1 then D(s, q) = 1.

The *truth value of a well-formed-formula (wff) A at a point* $t \in$ S, written $[\mathrm{A}]_t$, is defined by induction as follows:

1) $[A]_t = \mathrm{D}(t, A)$, for A atomic; $[f\,]_t = 0$.

2) $[A \wedge B]_t = 1$ iff $[A]_t = 1$ and $[B]_t = 1$.

3) $[A \vee B]_t = 1$ iff $[A]_t = 1$ or $[B]_t = 1$.

4) $[A \Rightarrow B]_t = 1$ iff for all s, if t Rs and $[A]_s = 1$, then $[B]_s = 1$.

[18] Definition 8 on p. 64 of [Gabbay 1981].

A structure is said to *validate* A iff $[A]_O = 1$. In this case A is also said to *hold* in the structure. (S, R, O) validates A iff for any D, (S, R, O, D) validates A.

Although the above approach to a Kripke structure is the more generally useful interpretation in a variety of logics, non-classical as well as classical, it is not the appropriate interpretation for situation theory. As is shown below, this is because of the peculiar nature of negation (the dual operator) and the partial nature of situations in situation theory. This difficulty is found when defining a truth valuation function, D, that honors the conditions of situation theory.

There are three approaches to the definition of the truth valuation function. One is to define D(t,q) = 1 iff t |= q, D(t,q) = 0, otherwise. The other two define $D(t, q) = 1$ iff ($t \models q$ and polarity of q is positive), they differ on the way to define $D(t, q) = 0$; either $D(t, q) = 0$ iff ($t \models q$ and polarity of q is negative), or $D(t, q) = 0$ iff $D(t, q) \neq 1$. The first approach incorrectly allows a situation to support both an infon *and its dual*; it allows $D(t,q) = D(t,\overline{q}) = 1$. The second approach gives a version of D that is partial, the latter one that is total, but doesn't distinguish between the support of the negative of an infon and the non-support of that infon (or its negative).The definition of a Kripke structure indicates that D must be total, so the first approach doesn't work. The third approach doesn't distinguish between significantly different circumstances, so it is not satisfactory either. Thus, none of these approaches is acceptable as models of situation theory.

There is an appropriate interpretation, however. It is known as the *strong Kripke propositional structure*. Its definition is given below.

A *strong propositional Kripke structure* has the form (S, R, O, α)[19], where (S, R, O) is a partially ordered set with a first element O and α is a function such that for each $t \in$ S and atomic q, $\alpha(t, q) \in \{-1, 0, 1\}$. (R is the relation that partially orders the elements of S. By the definition of a partially ordering relation, R is a transitive and reflexive relation.) If tRs and $\alpha(t, q) \neq 0$ then $\alpha(t, q) = \alpha(s, q)$.

[19] Definition 8 on p. 125 of [Gabbay 1981].

The *truth value of a well-formed-formula (wff) A at a point* $t \in$ S, written $[A]_t$, is defined by induction as follows:

1) $[A]_t = \alpha(t, A)$, for A atomic.
2) $[A \wedge B]_t = \min([A]_t, [B]_t)$.
3) $[A \vee B]_t = \max([A]_t, [B]_t)$.
4) $[A \Rightarrow B]_t = 1$ iff for all s, if t Rs and $[A]_s = 1$, then $[B]_s = 1$.
5) $[A \Rightarrow B]_t = -1$ iff $[A]_t = 1$ then $[B]_t = -1$.
6) $[-A]_t = 1$ iff $[A]_t = -1$.

A structure is said to *validate A* iff $[A]_O = 1$. In this case A is also said to *hold* in the structure. (S, R, O) validates A iff for any α, (S, R, O, α) validates A.

The definitions of the connectives $\wedge$, $\vee$, and$\Rightarrow$ are discussed in the next section. At this point, only the connective-free aspect of the Kripke structure is being examined. Non-parametric basic infons in situation theory can be described in terms of a propositional Kripke structure as follows: The atomic propositions of the structure are the non-parametric basic infons (of either polarity). The set[20] S is the set of all situations. The relation R is the "part of" relation between situations. Define O to be the minimal situation with respect to R that supports no infons. For technical reasons in dealing with quantification, it is convenient to define this minimal situation O to have one propertyless constituent.[21] The truth valuation function is defined by:

$\alpha(\mathrm{t}, q) = 1$ iff $t \models q$,

$\alpha(\mathrm{t}, q) = -1$ iff $t \models \overline{q}$,

$\alpha(\mathrm{t}, q) = 0$ iff $t \not\models q$ and $t \not\models \overline{q}$.

α is a function since only one of the three conditions can hold for a given pair of t and q. The persistence condition(Support condition 3, page 37) on the supports relation gives the property that if $\alpha(t, q) \neq 0$ and t Rs then $\alpha(t, q) = \alpha(s, q)$, as required by the definition of a strong propositional Kripke structure. This function is total,

[20] It may be a problem to consider S a *set*, unless one is using something like Aczel's ZFC-/AFA.

[21] Since O is, by definition, part of every situation, then the constituent of O can be a constituent of every situation.

and since no infon and its dual is supported by the same situation, $\alpha(t, q) = \alpha(t, \overline{q})$ iff $\alpha(t, q) = \alpha(t, \overline{q}) = 0$.

Given the above definition of a situation theoretic Kripke structure, no non-parametric basic infon is validated. That is, there is no basic infon that is supported by all situations, including the minimal situation.

5) The supports relation definition is extended with conditions defining the support of confirmation and denial of conjunction and disjunction.

The above discussion of a situation theory propositional Kripke structure can be extended to include non-parametric compound infons constructed using the connectives $\wedge$, $\vee$, and $\Rightarrow$. First, the conditions on the supports relation must be extended to include these compound infons. The notation here has a special interpretation. As noted above in the discussion of compound infons, the compound infon '$\sigma \wedge \tau$' is actually a short-hand notation for a *second-order* infon: $\langle\langle$'$\wedge$', $\sigma, \tau; +\rangle\rangle$. Similarly, '$\sigma \vee \tau$' is actually a short-hand notation for a *second-order* infon: $\langle\langle$'$\vee$', $\sigma, \tau; +\rangle\rangle$. These are second order infons since they take other infons as arguments. The notation for the duals of these infons is '$-(\sigma \wedge \tau)$' and '$-(\sigma \vee \tau)$', respectively. A presentation of the following interpretations of these connectives can be found on pp. 234-235 of [Barwise 1988].

Support condition 4. *Confirmation of Negation*: For all non-parametric infons σ, $s \models -\sigma$ iff $s \models \overline{\sigma}$. This is a simple claim that to say a situation confirms the second-order '$-\sigma$' infon is the same as claiming that situation supports the dual of σ.

Support condition 5. *Denial of Negation*: For all non-parametric infons σ, $s \models - - \sigma$ iff $s \models \sigma$. For a situation s to deny the second-order '$-\sigma$' infon is the same as claiming that s supports σ.

Support condition 6. *Confirmation of Conjunction*: For all non-parametric infons σ and τ, $s \models \sigma \wedge \tau$ iff $s \models \sigma$ and $s \models \tau$. This follows naturally from the idea of situations as part of the real world; the conjunction of two states-of-affairs is supported by some part of the real world if each of the states-of-affairs is individually supported.

Support condition 7. *Denial of Conjunction*: For all non-parametric infons σ and τ, $s \models -(\sigma \wedge \tau)$ iff $s \models \overline{\sigma}$ or $s \models \overline{\tau}$. This follows naturally from the idea of situations as part of the real world; the conjunction of two states-of-affairs is negated by some part of the real world if at least one of the states-of-affairs is individually negated (i.e. has its dual supported).

Support condition 8. *Confirmation of Disjunction*: For all non-parametric infons σ and τ, $s \models \sigma \vee \tau$ iff $s \models \sigma$ or $s \models \tau$. This also follows naturally from the idea of situations as part of the real world; the disjunction of two states-of-affairs is supported by some part of the real world if either or both of the states-of-affairs is individually supported.

Support condition 9. *Denial of Disjunction*: For all non-parametric infons σ and τ, $s \models -(\sigma \vee \tau)$ iff $s \models \overline{\sigma}$ and $s \models \overline{\tau}$. The disjunction of two states-of-affairs is denied by some part of the real world if both of the states-of-affairs is individually denied.

These additional definitions of the supports relation honor the original definitions as well. Particularly, it is true for compound infons as well as for basic infons that the support of an infon ‘persists’ from a situation to any containing situation. This is stated in the following theorem:

Theorem 1: All conditional-free infons are persistent.[22]

6) The conditional-free supports relation is shown to define a conditional-free strong propositional Kripke structure.

The α function of the Kripke structure is validly extended with these additional definitions for ‘supports’ by making q range over the set of *all* infons, instead of just the basic infons (corresponding to the atomic wffs). For this Kripke structure to be con-

[22] The IL (Infon Logic) theorems are proved in Appendix 1.

sistent with the 'supports' conditions, $[A]_t = \alpha(t, A)$ must hold for all conditional-free A. To aid in proving this theorem, it is useful to note that: If ($\alpha(t, A) = 1$ iff $[A]_t = 1$) and ($\alpha(t, A) = -1$ iff $[A]_t = -1$), then ($\alpha(t, A) = [A]_t$).

Theorem 2: For all wffs A and situations t, if ($\alpha(t, A) = 1$ iff $[A]_t = 1$) and ($\alpha(t, A) = -1$ iff $[A]_t = -1$), then ($\alpha(t, A) = [A]_t$).

The main theorem, that the conditional-free propositional supports relation defines a strong Kripke structure, follows:

Theorem 3: For all conditional-free propositional A, $[A]_t = \alpha(\mathrm{t}, A)$.

7) The Scott and Tarski consequence relations are defined.

To motivate the interpretation of the "conditional" connective '$\Rightarrow$', some background is required.[23] First, the idea of consequence is defined. If one is given a set of infons that are supported by some situation *s*, then there is some set of infons that can be inferred – that are also clearly supported. This is very generally to describe a kind of reasoning that people do constantly. This relationship between infons (or, traditionally, propositions) is called "consequence". The inferred infons are a consequence of the given infons. Dana Scott and Tarski have defined the properties that *any* formalism for a "consequence operator" should have. These follow from the informal understanding of consequence. They produced somewhat different formalizations, but there is a well understood relationship between these formalizations. Scott consequence operators are written ' ||-', and Tarski consequence operators are written '|-'. Scott consequence operators are a more general notion. They are defined as follows:

[23] [Barwise 1988], p. 184, introduces the conditional connective by simply giving an axiom for it - that it is reflexive and transitive, a pre-order. This axiom also states that *modus ponens* holds for infon conditional. It is interesting to note that *modus tolens* does *not* hold for the conditional connective defined here. Barwise provides too little information to determine if *modus tolens* holds for his notion of implication. This thesis argues for why the definitions in this thesis are the correct ones. Barwise does not define when an implication holds, which is a major undertaking of this subsection.

Let φ and ψ be finite, possibly empty, sets of well-formed formulae of the language of the operator, L. $\|\!-$ is a *Scott consequence relation* iff the following conditions hold[24]:

a) $\varphi \|\!- \varphi$, for $\varphi \neq \emptyset$.

b) if $\varphi \|\!- \psi$ then $\varphi \cup \varphi' \|\!- \psi \cup \psi'$ for any φ' and ψ'.

c) if $\varphi \cup \{A\} \|\!- \psi$ and $\varphi \|\!- \psi \cup \{A\}$ then $\varphi \|\!- \psi$. [Cut Rule]

Additional definitions:

For Δ and Θ sets of wffs, $\Delta \|\!- \Theta$ iff for some $\Delta \supseteq \varphi$ and $\Theta \supseteq \psi$, $\varphi \|\!- \psi$.

A Scott consequence relation is *consistent* iff $\sim(\emptyset \|\!- \emptyset)$.

Notational abbreviations:

$\varphi, A \|\!- \psi$ is the same as $\varphi \cup \{A\} \|\!- \psi$.

$\varphi,\varphi' \|\!- \psi,\psi'$ is the same as $\varphi \cup \varphi' \|\!- \psi \cup \psi'$.

$\varphi, A_1, A_2, \ldots, A_n \|\!- \psi$ is the same as $\varphi \cup \{A_1, A_2, \ldots, A_n\} \|\!- \psi$.

$|\!-$ is a *Tarski consequence relation* iff the following conditions hold[25]:

a) $A |\!- A$.

b) if $\varphi |\!- A$ then $\varphi, \varphi' |\!- A$.

c) if $\varphi, C |\!- A$ and $\varphi |\!- C$ then $\varphi |\!- A$. [Cut Rule]

The right-hand side argument of $|\!-$ is always a single wff.

The minimal meaning for '$\Rightarrow$' can be expressed by the *deduction theorem*:

$\varphi \cup \{A\} \|\!- \{B\}$ iff $\varphi \|\!- \{A \Rightarrow B\}$.

Let $\|\!-_{DT}$ be the minimal SCR for which the deduction theorem holds.

The Scott consequence relation (SCR) that is to be defined for situation theory infons must satisfy the deduction theorem, this is the minimal meaning accepted in this thesis for the '$\Rightarrow$' connective in infons – the '$\Rightarrow$' connective must satisfy *at least* this constraint.

[24] p. 6 of [Gabbay 1981].

[25] p. 7 of [Gabbay 1981].

An axiom (or Hilbert) system for some language L is defined by a triple (H_0, H_1, H_2)[26], where H_0 is a set of axiom schemas, H_1 is a set of provability rules, and H_2 is a set of consequence rules. An axiom schema (in H_0) is some wff in L, with propositional variables as one or more terms. A provability rule is of the form 'A1,…, An/B', where this rule is used in constructing a proof from the axiom schemas in H_0. A consequence rule is of the form {A1,…, An}/{B}, where this rule is used in constructing a proof of B given some set φ of wffs (and the things provable from the axioms). An axiom system can be used to define a Tarski consequence relation (TCR) '$\vdash_H$' as follows:

1) $\vdash_H A$ iff there exists a finite sequence of wff $B1,\dots,Bk = A$ such that each Bi of the sequence is either a substitution instance of a member of H_0, or for some wffs $A1,\dots,An$, appearing earlier than Bi in the sequence, $(A1,\dots,An/Bi)$ is a rule in H_1.

2) $\varphi \vdash_H A$ iff there exists a finite sequence of wffs $B1,\dots,Bn$ such that both (a) and (b) below hold:
   (a) For each $i \le n$, either (i), (ii), or (iii) below hold:
      i) $Bi \in \varphi$, or
      ii) $\vdash_H Bi$ (by 1 above), or
      iii) There exists $A1,\dots,Ak$ earlier in the sequence such that $\{A1,\dots,Ak\}/\{Bi\}$ is a substitution instance of a rule in H_2.
   (b) Either (i) or (ii) below hold:
      i) $A = B$, or
      ii) $\{B1,\dots,Bn\}/A$ is a substitution instance of a rule in H_2.

The relation $\vdash_H$ is proved by Gabbay to be a Tarski system for any axiom system H = (H_0, H_1, H_2). Further, he proves that for any Tarski system $\vdash$ there is an axiom

[26] p. 9 of [Gabbay 1981].

system H such that $|\!- = |\!-_H$.

8) The supports relation conditions for implication are developed with regard to a “minimal” concept of implication with respect to the Scott consequence relation.

Now to get back to the problem of the ‘⇒’ connective. Define an axiom system H^{++} as follows:

$H^{++}_0 = \{A \Rightarrow (B \Rightarrow A), [A \Rightarrow (B \Rightarrow C)] \Rightarrow [(A \Rightarrow B) \Rightarrow (A \Rightarrow C)]\}$.

$H^{++}_1 = \{(A, A \Rightarrow B/B)\}$

$H^{++}_2 = \{(\{A, A \Rightarrow B\}/\{B\})\}$

Gabbay proves the following equivalence: $|\!-_{H++}\ A$ iff $\varnothing\ |\!|\!-_{DT}\ \{A\}$[27]. This says that a wff A is a Tarski consequence of the H^{++} axiom system if and only if A is a tautology of the $|\!|\!-_{DT}$ SCR, which is the smallest SCR for which the deduction theorem holds. H^{++} is known as the conditional fragment of Heyting’s propositional calculus.

This discussion leads to the conclusion that infon logic includes the axioms of H^{++}. The infon logic axiom system includes the simple axioms of ‘∨’ and ‘∧’ with ‘⇒’, which gives the propositional fragment of Heyting’s Predicate Calculus (HPC) Gabbay calls *h*-.

The axiom system for *h* is[28]:

$h_0 = \{$

(a) $A \Rightarrow (B \Rightarrow A)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,

(c) $A \wedge B \Rightarrow A,\ A \wedge B \Rightarrow B$,

(d) $A \Rightarrow (B \Rightarrow A \wedge B)$,

[27] p. 23 in [Gabbay 1981].
[28] p. 63 in [Gabbay 1981].

(e) $A \Rightarrow (A \vee B)$, $B \Rightarrow (A \vee B)$,

(f) $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))$,

(g) $\boldsymbol{f} \Rightarrow A$ }

h_1 = {(A, A⇒B/B)}

h_2 = {({A, A⇒B}/{B})}

h_1 and h_2 are the same as for H^{++}, and define modus ponens.

The axiom system h^- is h without axiom ‘g’. h^- is the negation-free propositional fragment of HPC.

9) Strong negation axioms are introduced into the h system to give the nh system.

The strong negation Heyting propositional axiom system, called *nh*, is h^- plus the following axioms for negation:

0) $--A \Leftrightarrow A$.

1) $-(A \Rightarrow B) \Leftrightarrow A \wedge -B$.

2) $-(A \wedge B) \Leftrightarrow -A \vee -B$.

3) $-(A \vee B) \Leftrightarrow -A \wedge -B$.

4) $A \wedge -A \Rightarrow B$.

nh is H^{++} plus axioms for $\wedge$, $\vee$ and $-$. These axioms are presented by Gabbay[29] stating “Some authors introduced another type of negation into HPC, called strong negation.” Unfortunately, he does not identify any of these strong-negation-introducing authors.

The important result here is that the Scott consequence system defined from the strong propositional Kripke structure has exactly the same theorems as the Tarski consequence system defined from *nh*. Thus, *nh* is a minimal appropriate axiom system for the strong propositional Kripke structure. Since the conditional-free supports relation conditions can be used to define a (conditional-free) strong propositional

[29] p. 124 of [Gabbay 1981].

Kripke structure, and the *nh* conditional axioms are just those desired for infon implication, then extending the supports relation for the conditional connective in such a way as to allow definition of a full strong propositional Kripke structure provides the desired idea of implication in infon logic.

The extension to the supports relation implied by the foregoing is:

Support condition 10. *Confirmation of Conditional*: For all non-parametric infons σ and τ, $s \models \sigma \Rightarrow \tau$ iff for all t such that $s \leq_S t$, $t \models \sigma$ implies $t \models \tau$.

Support condition 11. *Denial of Conditional*: For all non-parametric infons σ and τ, $s \models -(\sigma \Rightarrow \tau)$ iff $s \models \sigma$ and $s \models \bar{\tau}$.

The persistence of conditional-free infons can be extended to include conditional infons. Thus, all propositional infons can be shown to be persistent.

Theorem 4: Propositional infons are persistent.

To claim that the set of infons a situation supports contains a conditional infon is also to claim that that set of infons is closed with respect to that conditional infon. This is based on Support condition 10.

10) The propositional supports relation is shown to define a strong propositional Kripke structure.

The proof that $\alpha(t, P) = [P]_t$ for all conditional-free infons P can be extended to include the '$\Rightarrow$' connective by proving that $\alpha(t, A \Rightarrow B) = [A \Rightarrow B]_t$, if $\alpha(t, A) = [A]_t$ and $\alpha(t, B) = [B]_t$.

Theorem 5: If $\alpha(t, A) = [A]_t$ and $\alpha(t, B) = [B]_t$, then $\alpha(t, A \Rightarrow B) = [A \Rightarrow B]_t$.

11) The propositional fragment with strong negation of Heyt-

ing's predicate calculus is shown to be the "supports-preserving" axiom system for the propositional supports relation.

The above discussion establishes that the axiom system for situation theory quantifier-free infon logic is Heyting's propositional logic with strong negation (*nh*), and that the definition of the "supports" relation is as given in Support postulate 1, Support postulate 2, and Support condition 1 through Support condition 11.

12) The quantificational conditions for the supports relation are given.

The remaining problem for infon logic is to define the meaning of quantification in infon logic, and to axiomatize this meaning. First, some notation for dealing with parameters:

$\boldsymbol{f}_t = \{\mathbf{x}/\mathrm{a}, \mathbf{y}/\mathrm{b}, ...\}$ is an *anchor*, a set of assignments of parameters to "values" (which may themselves be parameters). $\sigma[\boldsymbol{f}_t]$ is the infon produced by "applying" the anchor $\boldsymbol{f}_t$ to the infon σ. Applying an anchor consists of substituting all of the left-hand side parameters of the pairs in the anchor with the associated right-hand side item.

A *parametric infon* is an infon that has one or more free parameters. A parameter is free in an infon if it is not used as the first argument to a quantifier (in that infon). A parametric infon is written as '$\sigma(\boldsymbol{x})$', where σ is any infon and the parameter $\boldsymbol{x}$ is free in σ.

This thesis defines the interpretation of quantifiers in a manner different from the situation theory literature. On p. 235 of [Barwise 1988], confirmation of existential quantification is done in the same way, but the negation is handled differently. Barwise foregoes the persistence postulate to have a very simple definition of the negation of existential quantification: $s \models -\exists \mathbf{x}\sigma$ iff for all non-parametric *anchorings* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in$ *constituents*(s) $s \models -\sigma[\boldsymbol{f}]$. But, for an only marginally more complex notion of the denial of existential quantification, the persistence postulate can be saved.

Also, if one accepts this somewhat more complex definition the existential and universal quantifiers are naturally duals. The approach used here is more general, in that it can be restricted to have the same meaning as that used by Barwise by the addition of more conjuncts to the quantified infon.

The quantificational supports conditions are as follows:

Support condition 12. *Confirmation of Universal Quantification*: For all infons σ parametric only in $\boldsymbol{x}$, $s \models \forall \mathbf{x}\sigma$ iff for all situations t and non-parametric *anchorings* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in constituents(t)$, $s \leq_S t$ implies $t \models \sigma[\boldsymbol{f}]$.

Support condition 13. *Denial of Universal Quantification*: For all infons σ parametric only in $\boldsymbol{x}$, $s \models -\forall \mathbf{x}\sigma$ iff there exists a non-parametric *anchoring* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in constituents(s)$, such that $s \models -\sigma[\boldsymbol{f}]$.

Support condition 14. *Confirmation of Existential Quantification*: For all infons σ parametric only in $\boldsymbol{x}$, $s \models \exists \mathbf{x}\sigma$ iff there exists some non-parametric *anchoring* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in constituents(s)$ such that $s \models \sigma[\boldsymbol{f}]$.

Support condition 15. *Denial of Existential Quantification*: For all infons σ parametric only in $\boldsymbol{x}$, $s \models -\exists \mathbf{x}\sigma$ iff for all situations t and non-parametric *anchorings* $\boldsymbol{f} = \{\mathbf{x}/a\}$, $a \in constituents(t)$, $s \leq_S t$ implies $t \models -\sigma[\boldsymbol{f}]$.

By these conditions, $s \models -\forall \mathbf{x}\sigma$ iff $s \models \exists \mathbf{x} -\sigma$, and $s \models -\exists \mathbf{x}\sigma$ iff $s \models \forall \mathbf{x} -\sigma$. Thus, universal and existential quantification are "dual" operators with respect to negation in situation theory infon logic, as they are in classical logic.

If $s \models \forall \mathbf{x}\sigma$ and $s \leq_S t$, then for all $\boldsymbol{f}_t$, $t \models \sigma[\boldsymbol{f}_t]$, by Support condition 12. Let r be any situation such that $t \leq_S r$. Since $s \leq_S r$ (by transitivity), then for all $\boldsymbol{f}_r$, $r \models \sigma[\boldsymbol{f}_r]$. Thus, for all r such that $t \leq_S r$, for all $\boldsymbol{f}_r$, $r \models \sigma[\boldsymbol{f}_r]$. $t \models \forall \mathbf{x}\sigma$ if for all r such that $t \leq_S r$, for all $\boldsymbol{f}_r$, $r \models \sigma[\boldsymbol{f}_r]$, by Support condition 12. Therefore, if $s \models \forall \mathbf{x}\sigma$ and $s \leq_S t$, then $t \models \forall \mathbf{x}\sigma$. Thus, $\forall \mathbf{x}\sigma$ is a persistent infon. A similar argument can be made to show that the dual of $\forall \mathbf{x}\sigma$, $-\exists \mathbf{x}\sigma$, is persistent.

If $s \models \exists \mathbf{x}\sigma$ then there exists $\boldsymbol{f}_s$ such that $s \models \sigma[\boldsymbol{f}_s]$, by Support condition 14. If $s \leq_S t$, then if $s \models \sigma[\boldsymbol{f}_s]$, then $t \models \sigma[\boldsymbol{f}_s]$ by persistence of infons (there is a possible recursion here - assume that $\sigma[\boldsymbol{f}_s]$ is not quantified). There exists $\boldsymbol{f}_t$ such that $\boldsymbol{f}_s = \boldsymbol{f}_t$, since $s \leq_S t$. Thus, if $t \models \sigma[\boldsymbol{f}_s]$ then $t \models \sigma[\boldsymbol{f}_t]$. If $t \models \sigma[\boldsymbol{f}_t]$, then $t \models \exists \mathbf{x}\sigma$, by Support condition 14. Therefore, If $s \models \exists \mathbf{x}\sigma$ and $s \leq_S t$, then $t \models \exists \mathbf{x}\sigma$. Thus, $\exists \mathbf{x}\sigma$ is a persistent infon. A similar argument can be made for $-\forall \mathbf{x}\sigma$.

By the preceding arguments, the quantified infons are persistent.

13) Heyting's Predicate Calculus, axiom system H, is presented.

The axiom system H, Heyting's Predicate Calculus (HPC), is an extension of *h*:

$H_0 = h_0 \cup \{$

(h) $A(x) \Rightarrow \exists y\, A(y)$,

(k) $\forall y\, A(y) \Rightarrow A(x)\}$

$H_1 = h_1 \cup \{$

$(A(x) \Rightarrow B \,/\, \exists x A(x) \Rightarrow B)$,

$(B \Rightarrow A(x) \,/\, B \Rightarrow \forall x A(x))\}$

$H_2 = h_2$.

14) The (quantified) Kripke structure is introduced.

A *Kripke structure*[30] is an extension of the propositional Kripke structure to handle variables and quantification. Kripke structures have the form (N, S, R, O, D, U), where $S \supseteq N$, $S^2 \supseteq R$, $O \in S$, and the following hold:

(a) R is a reflexive and transitive relation on S.

(b) O Rx for all $x \in S$.

(c) $t \in N$ and tRs imply $s \in N$.

(d) U is a function associating with each $t \in S$ a nonempty set U_t and if sRs' then $U_{s'} \supseteq U_s$.

(e) D is a function such that for each n-place atomic A, and each t, $U_t^n \supseteq D(t, A)$. D has the property that if tRs then $D(s, A) \supseteq D(t, A)$, for all atomic A. If $t \in N$ then $D(t, A) = U_t^n$.

Let g:V -> U_t, $t \in S$ (V the set of variables of H). Define the truth value of A, $[A]_t^g$,

[30] p. 43 in [Gabbay 1981].

by induction on A as follows[31]:

(a) $[A(x_1, \ldots, x_n)]_t^g = 1$ iff $(g(x_1), \ldots, g(x_n)) \in \mathrm{D}(t, A)$ if A is atomic with $x_1, \ldots, x_n$ free in A.

(b) $[A \wedge B]_t^g = 1$ iff $[A]_t^g = 1$ and $[B]_t^g = 1$.

(c) $[A \vee B]_t^g = 1$ iff $[A]_t^g = 1$ or $[B]_t^g = 1$.

(d) $[\boldsymbol{f}\,]_t^g = 1$ iff $\boldsymbol{f} \in \mathrm{N}$.

(e) $[A \Rightarrow B]_t^g = 1$ iff for all s, if $t\mathrm{R}s$ and $[A]_s^g = 1$ then $[B]_s^g = 1$.

(f) $[\exists x A(x)]_t^g = 1$ iff for some $g' =_x g$, $[A(x)]_t^{g'} = 1$ where $g' =_x g$ means that for all $y \neq x$, $g(y) = g'(y)$.

(g) $[\forall x A(x)]_t^g = 1$ iff for all s, g' (if $t\mathrm{R}s$ and $g' =_x g$ then $[A(x)]_s^{g'} = 1$).

(h) A is said to hold in the structure under g iff $[A]_O^g = 1$.

Gabbay[32] defines a Scott consequence relation defined on a *class* of Kripke structures. He shows that all of the tautologies of this SCR are exactly the theorems of Heyting's Predicate Calculus (HPC), which is identified with the axiom system H.

15) The strong Kripke structure is defined.

Strong Kripke structures are a combination of quantified Kripke structures and strong propositional Kripke structures.[33] They have the form (N, S, R, O, α, U), where $\mathrm{S} \supseteq \mathrm{N}$, $\mathrm{S}^2 \supseteq \mathrm{R}$, $\mathrm{O} \in \mathrm{S}$, and the following hold:

(a) R is a reflexive and transitive relation on S.

(b) $\mathrm{OR}x$ for all $x \in \mathrm{S}$.

(c) $t \in \mathrm{N}$ and $t\mathrm{R}s$ imply $s \in \mathrm{N}$.

(d) U is a function associating with each $t \in \mathrm{S}$ a nonempty set U_t, and if $s\mathrm{R}s'$ then $\mathrm{U}_{s'} \supseteq \mathrm{U}_s$.

[31] p. 44 in [Gabbay 1981].

[32] p.46 in [Gabbay 1981].

[33] This structure is original to this thesis.

(e) α is a function such that for each n-place atomic A, and each t, $U_t^n \supseteq \alpha(t, A, k)$. α has the property that if $t\mathrm{R}s$ then $\alpha(s, A, k) \supseteq \alpha(t, A, k)$, for all atomic A and $k \in \{1, -1\}$. If $t \in \mathrm{N}$ then $\alpha(t, A, 1) = U_t^n$ and $\alpha(t, A, -1) = \varnothing$. (The degenerate case for α is when $n = 0$. In this case $\alpha(t, A, k)$ is either empty or contains the empty set.)

Let g:V -> U_t, $t \in$ S (V the set of variables of NH). Define the truth value of A, $[A]_t^g$, by induction on A as follows[34]:

(1) $[A(x_1, \ldots, x_n)]_t^g = \mathrm{k}$ iff $(g(x_1), \ldots, g(x_n)) \in \alpha(t, A, \mathrm{k})$ if A is atomic with $x_1, \ldots, x_n$ free in A, $\mathrm{k} \in \{-1, 1\}$.

(2) $[A \wedge B]_t^g = \min([A]_t^g, [B]_t^g)$.

(3) $[A \vee B]_t^g = \max([A]_t^g, [B]_t^g)$.

(4) $[A \Rightarrow B]_t^g = 1$ iff for all s, if $t\mathrm{R}s$ and $[A]_s^g = 1$ then $[B]_s^g = 1$.

$[A \Rightarrow B]_t^g = -1$ iff $[A]_t^g = 1$ and $[B]_t^g = -1$.

(5) $[-A]_t^g = 1$ iff $[A]_t^g = -1$.

(6) $[\exists x A(x)]_t^g = 1$ iff for some $g' =_x g$, $[A(x)]_t^{g'} = 1$ where $g' =_x g$ means that for all $y \neq x$, $g(y) = g'(y)$.

$[\exists x A(x)]_t^g = -1$ iff for all s, g' (if $t\mathrm{R}s$ and $g' =_x g$ then $[A(x)]_s^{g'} = -1$).

(7) $[\forall x A(x)]_t^g = 1$ iff for all s, g' (if $t\mathrm{R}s$ and $g' =_x g$ then $[A(x)]_s^{g'} = 1$).

$[\forall x A(x)]_t^g = -1$ iff for some $g' =_x g$, $[A(x)]_t^{g'} = -1$.

(8) A is said to hold in the structure under g iff $[A]_O^g = 1$.

The axiom system for infon logic is NH, which is H with the axiom for $\boldsymbol{f} \Rightarrow A$ replaced by the axioms for strong negation, 0 through 4 given above, plus axioms for negation of quantifiers:

(5) $-\exists x A(x) \Leftrightarrow \forall x -A(x)$,

(6) $-\forall x A(x) \Leftrightarrow \exists x -A(x)$.

[34] p. 44 in [Gabbay 1981].

The theorems of NH are the tautologies of the SCR defined on the class of Kripke structures which consists of the strong Kripke structure based on situation theory. The TCR for NH is written ‘ $|\!-_{NH}$’.

16) The full supports relation is shown to define a strong Kripke structure.

The SCR for the class of Kripke structures which consists of the situation theory infon logic Kripke structure is written ‘$\|\!-_{ST}$’.

A strong Kripke structure can be defined for situation theory and the supports condition as follows: Let R be the ‘part of’ relation. Let O be the empty situation. Let N = Ø. Let U be the *constituents* function, where *constituents*(t) = U_t. Let $(x_1, \ldots, x_n)$ be the free parameters in A and $\boldsymbol{f}$ = {x1/a1,…, xn/an}. Define $(a_1, \ldots, a_n) \in \alpha(t, A, 1)$ iff $t \models A[\boldsymbol{f}]$, and $(a_1, \ldots, a_n) \in \alpha(t, A, -1)$ iff $t \models \overline{A}[\boldsymbol{f}]$. Let ‘()’ denote the empty tuple (the type of 0 arity). If A has no free parameters, then $() \in \alpha(t, A, 1)$ iff $t \models A$, and $() \in \alpha(t, A, -1)$ iff $t \models \overline{A}$.

The given definition for α applies to all infons A, not only basic infons. For this definition to be consistent with the strong Kripke structure, $[A(x_1, \ldots, x_n)]_t^g = k$ iff $(g(x_1), \ldots, g(x_n)) \in \alpha(t, A, k)$.

Conjecture: For all infons A, situations t, and anchors g, $[A(x_1, \ldots, x_n)]_t^g = k$ iff $(g(x_1), \ldots, g(x_n)) \in \alpha(t, A, k)$.

The proof of this conjecture may be done in a manner similar to that used above for the strong propositional Kripke structure.

17) The NH axiom system is proposed as the supports-preserving axiom system for the full supports relation.

The following conjecture guarantees that results derived in the formal axiom system NH are semantically valid with respect to situation theory infon logic.

Conjecture: $|\!-_{NH}\ A$ iff $\varnothing \,||\!-_{ST}\ \{A\}$

The proof is in two parts: a proof of $|\!-_{NH}\ A$ implies $\varnothing \,||\!-_{ST}\ \{A\}$, and a proof of $|\!-_{NH}\ A$ if $\varnothing \,||\!-_{ST}\ \{A\}$. The difficulty lies in ST being a strong Kripke structure, not simply a Kripke structure, and the ordering relation R of ST is *not* isomorphic to a tree ordering relation. This latter point is unfortunate since the strongest results Gabbay proves are for "tree" Kripke structures.

Semi-Proof of $|\!-_{NH}\ A$ implies $\varnothing \,||\!-_{ST}\ \{A\}$:

Let '$\varphi ||\!-_{nh} \psi$' be the SCR defined by '$\wedge\varphi \,|\!-_{nh} \vee\psi$' ($\wedge\varnothing = (p \Rightarrow p)$, $\vee\varnothing = (p \wedge -p)$).

$||\!-_{nh}$ agrees with $|\!-_{nh}$, by Proposition 11.a on p. 125 of [Gabbay 1981].

$||\!-_{nh}$ is identical with the Scott consequence system arising from the interpretation in the strong propositional Kripke structures, by Theorem 12 on p. 125 of [Gabbay 1981]. Also, $\wedge$ and $\vee$ are *classical* in $||\!-$. In particular, $|\!-_{nh}\ A$ iff A is valid in every such structure.

Thus, $|\!-_{nh}\ A$ iff A is valid in the strong propositional Kripke structure for situation theory infon logic presented above.

Since A is valid with respect to structure for situation theory infon logic iff A is supported by the supports relation as defined in conditions 0 through 11, $|\!-_{nh}\ A$ iff A is supported by the supports relation as defined in conditions 0 through 11.

Let f be any fixed proposition in *nh* and let $\sim A$ in *nh* be '$A \Rightarrow (f \wedge -f)$', $\sim A$ in h is 'A => $\boldsymbol{f}$ ' ($\boldsymbol{f}$ is the defined 'false' symbol in h). From exercise 15 of p. 126 in [Gabbay 1981], for any wff B in h, *nh* $|\!-\ B$ iff $h\ |\!-\ B$. Thus, *nh* is a conservative extension of h.

Conjecture: NH is a conservative extension of H. This is based on the observation that *nh* is conservative extension of *h*.

Conjecture: The situation theory infon logic *strong Kripke structure* validates A iff $\vdash_{\mathrm{NH}} A$.

Theorem 6: If $s \models A$, and $A \vdash_{\mathrm{NH}} B$, then $s \models B$. (I.e., $\vdash_{\mathrm{NH}}$ is *sound* with respect to the $\models$ relation).
Proof:
To prove this theorem, it is sufficient to show that for all s, $s \models A$, for all A that are axioms in NH_0, and to show that the provability and consequence rules (NH_1 and NH_2) preserve the support relation.

It has already been shown that for all s, $s \models A$, for all A in *nh*0. Thus, to complete the demonstration for NH_0 it is only necessary to show that for all s, $s \models A$, for A in $\{A(x) \Rightarrow \exists y\, A(y), \forall y\, A(y) \Rightarrow A(x), -\exists x A(x) \Leftrightarrow \forall x\, -A(x), -\forall x A(x) \Leftrightarrow \exists x\, -A(x)\}$.

Since nh_1 has been shown to be support preserving, to complete the proof for NH_1 it is only necessary to show that $\{(A(x) \Rightarrow B\, /\, \exists x A(x) \Rightarrow B), (B \Rightarrow A(x)\, /\, B \Rightarrow \forall x A(x))\}$ (x not free in B) are support preserving.

Since nh_2 has been shown to be support preserving and $\mathrm{NH}_2 = nh_2$, NH_2 is support preserving.

Proof for axioms in NH_0 - nh_0:

Theorem 7: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of x to the constituents of s, $s \models (A(x) \Rightarrow \exists y\, A(y))[\boldsymbol{g}_s]$.

Theorem 8: For all s and non-parametric $\boldsymbol{g}_s$, $s \models (\forall y\, A(y) \Rightarrow A(x))[\boldsymbol{g}_s]$.

Theorem 9: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (-\exists x A(x) \Leftrightarrow \forall x\, -A(x))$.

Theorem 10: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (-\forall x A(x) \Leftrightarrow \exists x\, -A(x))[\boldsymbol{g}_s]$.

Proof for NH_1 - nh_1:
Let A be an axioms or a theorem derived from the axioms. Let B be any infon where x is not free in B.

Theorem 11: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (A(x) \Rightarrow B\,)[\boldsymbol{g}_s]$ implies for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (\exists x A(x) \Rightarrow B)[\boldsymbol{g}_s]$.

Theorem 12: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (B \Rightarrow A(x))[\boldsymbol{g}_s]$ implies for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (B \Rightarrow \forall x A(x))[\boldsymbol{g}_s]$.

This concludes the development of the set of support conditions and the infon axiom system NH.

Other Formalizations of the Infon Logic

The infon logic is formalized above as the Hilbert axiom system NH (Heyting's predicate calculus with strong negation). This axiomatic approach is convenient in the context of the overall approach taken to developing NH. However, other formalizations are more convenient under other circumstances. Two closely related such formalizations are natural deduction systems and Gentzen sequent calculus systems. Michael Dummett presents natural deduction and Gentzen sequent calculus formalizations of intuitionistic logic[35] (axiomatized as H in the preceding discussion), called N and L respectively. The strong negation extension of the natural deduction formalization, NN, is used in the development of a theorem prover for infon logic, FELIX, presented in a later section of the thesis.

Natural Deduction system for infon logic:

A natural deduction system can be constructed from Dummett's natural deduction system[36] for H. This system is called NN. There are no axioms in a natural deduction system. It consists entirely of inference rules. An inference rule is of the form

$$\frac{\Gamma : A}{\Gamma : B}$$

where '$\Gamma : A$' is the premise and '$\Gamma : B$' is the conclusion of the inference. '$\Gamma : A$' is read as "the set of wffs Γ *derives A*". The inference rule is read as "if Γ derives *A*, then Γ derives *B*." In some of the inference rules it is convenient to write "the union of the set of wffs Γ and the set of wffs Γ" as 'Γ, Δ'.

There are two kinds of inference rules, operator introduction rules and operator elimination rules. For most operators there is a pair of sets of rules - the introduction rule set and the elimination rule set. For most of the pairs of rule sets, each set consists of only one rule. There are several sets of rules for strong negation, one set for each combination of strong negation ('–') and another operator. Also, rules are provided defining the weak negation operator, '$\neg$ '.

Basic Sequent

[35] [Dummett 1977]

[36] pp. 123-124 of [Dummett 1977].

$\Gamma, A : A$ [$A : A$ always holds.]

Thinning Rule

$$\frac{\Gamma : B}{\Gamma, A : B}$$

Operator	*Introduction Rules*	*Elimination Rules*
$\wedge$	$\dfrac{\Gamma : A \qquad \Delta : B}{\Gamma, \Delta : A \wedge B}$	$\dfrac{\Gamma : A \wedge B}{\Gamma : A}$ $\qquad$ $\dfrac{\Gamma : A \wedge B}{\Gamma : B}$
$\vee$	$\dfrac{\Gamma : A}{\Gamma : A \vee B}$ $\qquad$ $\dfrac{\Gamma : B}{\Gamma : A \vee B}$	$\dfrac{\Gamma : A \vee B \qquad \Delta, A : C \qquad \Theta, B : C}{\Gamma, \Delta, \Theta : C}$
$\Rightarrow$	$\dfrac{\Gamma, A : B}{\Gamma : A \Rightarrow B}$	$\dfrac{\Gamma : A \qquad \Delta : A \Rightarrow B}{\Gamma, \Delta : B}$
$\neg$	$\dfrac{\Gamma, A : B \qquad \Delta, A : \neg B}{\Gamma, \Delta : \neg A}$	$\dfrac{\Gamma : A \qquad \Delta : \neg A}{\Gamma, \Delta : B}$
$-$		$\dfrac{\Gamma : - A}{\Gamma : \neg A}$
$- -$	$\dfrac{\Gamma : A}{\Gamma : - - A}$	$\dfrac{\Gamma : - - A}{\Gamma : A}$
$- \neg$	$\dfrac{\Gamma : A}{\Gamma : - \neg A}$	$\dfrac{\Gamma : - \neg A}{\Gamma : A}$
$- \Rightarrow$	$\dfrac{\Gamma : A \wedge - B}{\Gamma : -(A \Rightarrow B)}$	$\dfrac{\Gamma : -(A \Rightarrow B)}{\Gamma : A \wedge - B}$
$- \wedge$	$\dfrac{\Gamma : - A \vee - B}{\Gamma : -(A \wedge B)}$	$\dfrac{\Gamma : -(A \wedge B)}{\Gamma : - A \vee - B}$
$- \vee$	$\dfrac{\Gamma : - A \wedge - B}{\Gamma : -(A \vee B)}$	$\dfrac{\Gamma : -(A \vee B)}{\Gamma : - A \wedge - B}$

	$\dfrac{\Gamma : A(\mathrm{t})}{\Gamma : \exists \mathrm{x} A(\mathrm{x})}$	$\dfrac{\Gamma : \exists \mathrm{x} A(\mathrm{x}) \qquad \Delta, A(\mathrm{y}) : C}{\Gamma, \Delta : C}$
$\exists$		
	$\dfrac{\Gamma : A(\mathrm{y})}{\Gamma : \forall \mathrm{x} A(\mathrm{x})}$	$\dfrac{\Gamma : \forall \mathrm{x} A(\mathrm{x})}{\Gamma : A(\mathrm{t})}$
$\forall$		
	$\dfrac{\Gamma : \forall x -A(x)}{\Gamma : -\exists x A(x)}$	$\dfrac{\Gamma : -\exists x A(x)}{\Gamma : \forall x -A(x)}$
$-\exists$		
	$\dfrac{\Gamma : \exists x -A(x)}{\Gamma : -\forall x A(x)}$	$\dfrac{\Gamma : -\forall x A(x)}{\Gamma : \exists x -A(x)}$
$-\forall$		

For the quantification rules, the following additional constraints must hold:

1) y is a variable and t is any term of NH, where x is not free in t or y.
2) A(y) and A(t) result from A(x) by replacing every free occurrence of x by y and t respectively.
3) In the $\forall$ introduction rule, y does not occur free in $\Gamma : \forall \mathrm{x} A(\mathrm{x})$.
4) In the $\exists$ elimination rule, y does not occur free in $\exists \mathrm{x} A(\mathrm{x})$ or $\Gamma, \Delta : C$.

In the above table, there is a gap where one expects the '–' introduction rule. This is expressive of a difference between strong negation, '–', and weak negation, '¬'. The rule for weak negation introduction says roughly that if Γ and A derive B and Δ and A derive the weak negation of B, then the weak negation of A is derivable from Γ and Δ. With strong negation, '–', it is possible that Γ and Δ support (derive) *neither* B nor $-B$. Thus, there isn't a corresponding rule for strong negation introduction. Weak negation is defined in infon logic as $\neg A =_{\mathrm{df}} (A \Rightarrow \perp)$. The special infon $\perp$ (read "bottom") is unsupported by all situations, which is equivalent to $B \wedge - B$, for all B. The dual of this infon ('T' read "top"), is supported by all situations (including the origin situation O in the Kripke interpretations).

The weak negation introduction rule allows the use of a form of *reductio ad absurdam* reasoning: "if from A together with other hypotheses Γ we can derive an inconsistent pair of formulae B_1 and B_2, then we are entitled to assert $\neg A$ on the basis of Γ."[37]

Dummett points out that the system he gives is redundant. The thinning rule can be achieved from application of the '$\wedge$' introduction rule followed by an application of the '$\wedge$' elimination rule. Conversely, in the presence of the thinning rule, those rules with more than one premise can be weakened by writing 'Γ' in place of 'Δ' and 'Θ'. Also, the thinning rule allows the basic sequent to be defined in the more restricted form of '$A : A$'. Given the more general form of the basic sequent, the thinning rule can eliminated from the system and still use the restricted ('Γ' only) form of the rules.

[37] p. 125 in [Dummett 1977].

Gentzen Sequent Calculus system for NH.

Dummett provides a sequent calculus system L for intuitionistic logic (equivalent to the logic axiomatized by H)[38]. The system presented here, NL, is L extended with strong negation. For the sequent calculus system, a sequent is '$\Gamma : A$' or '$\Gamma :$', where 'Γ' is a set of formulæ and A is a single formula. The latter form of a sequent indicates that the antecedent ('Γ') is inconsistent. The kinds of rules in a sequent calculus system involve introduction on the left versus introduction on the right of the ':' symbol, instead of introduction versus elimination as was the case in the natural deduction system.

Operator	*Right Introduction*	*Left Introduction*
Thin	$\dfrac{\Gamma :}{\Gamma : A}$	$\dfrac{\Gamma : C}{\Gamma, A : C}$
$\wedge$	$\dfrac{\Gamma : A \qquad \Delta : B}{\Gamma, \Delta : A \wedge B}$	$\dfrac{\Gamma, A, B : C}{\Gamma, A \wedge B : C}$
$\vee$	$\dfrac{\Gamma : A}{\Gamma : A \vee B} \qquad \dfrac{\Gamma : B}{\Gamma : A \vee B}$	$\dfrac{\Gamma, A : C \qquad \Gamma, B : C}{\Gamma, \Delta, A \vee B : C}$
$\Rightarrow$	$\dfrac{\Gamma, A : B}{\Gamma : A \Rightarrow B}$	$\dfrac{\Gamma, B : C \qquad \Delta : A}{\Gamma, \Delta, A \Rightarrow B : C}$
$-$	$\dfrac{\Gamma,\ A :}{\Gamma : -A}$	$\dfrac{\Gamma : A}{\Gamma, -A :}$

[38] p. 135-137 in [Dummett 1977]. Dummett also presents a version of the sequent calculus formalization where the right hand side is a *set* of formulae instead of a single formula. Many of the inference rules translate simply by adding a 'set' variable to the RHS, but some rules (conditional and universal) must still be restricted to having a single formula on the RHS.

	$\Gamma : A$	$\Gamma , A : B$
$--$	$\overline{\Gamma : --A}$	$\overline{\Gamma, --A : B}$
	$\Gamma : A \wedge -B$	$\Gamma, A \wedge -B : C$
$- \Rightarrow$	$\overline{\Gamma : -(\mathrm{A} \Rightarrow B)}$	$\overline{\Gamma, -(\mathrm{A} \Rightarrow B) : C}$
	$\Gamma : -A \vee -B$	$\Gamma, -A \vee -B : C$
$- \wedge$	$\overline{\Gamma : -(A \wedge B)}$	$\overline{\Gamma, -(A \wedge B) : C}$
	$\Gamma : -A \wedge -B$	$\Gamma, -A \wedge -B : C$
$- \vee$	$\overline{\Gamma : -(A \vee B)}$	$\overline{\Gamma, -(A \vee B) : C}$
	$\Gamma : A(\mathrm{t})$	$\Gamma, A(\mathrm{y}) : C$
$\exists$	$\overline{\Gamma : \exists \mathrm{x} A(\mathrm{x})}$	$\overline{\Gamma, \exists \mathrm{x} A(\mathrm{x}) : C}$
	$\Gamma : A(\mathrm{y})$	$\Gamma, A(\mathrm{t}) : C$
$\forall$	$\overline{\Gamma : \forall \mathrm{x} A(\mathrm{x})}$	$\overline{\Gamma, \forall \mathrm{x} A(\mathrm{x}) : C}$
	$\Gamma : \forall x - A(x)$	$\Gamma, \forall x - A(x) : C$
$-\exists$	$\overline{\Gamma : -\exists x A(x)}$	$\overline{\Gamma, -\exists x A(x) : C}$
	$\Gamma : \exists x - A(x)$	$\Gamma, \exists x - A(x) : C$
$-\forall$	$\overline{\Gamma : -\forall x A(x)}$	$\overline{\Gamma, -\forall x A(x) : C}$

In all cases, ‘*C*’ is either a formula or the empty set. For the quantification rules, the following additional constraints must hold (these are the same as for the natural deduction system given above):

1) y is a variable and t is any term of NH, where x is not free in t or y.
2) A(y) and A(t) result from A(x) by replacing every free occurrence of x by y and t respectively.
3) In the $\forall$ right-introduction rule, y does not occur free in $\Gamma : \forall \mathrm{x} A(\mathrm{x})$.
4) In the $\exists$ left-introduction rule, y does not occur free in $\exists \mathrm{x} A(\mathrm{x})$ or $\Gamma, \exists \mathrm{x} A(\mathrm{x}) : C$.

Consider the sequent Γ, A(x):A(y), where x and y are free variables and A(x) is identical to A(y) except that all free occurrences of x in A(x) have been replaced by y in A(y). This sequent is *not* necessarily true (i.e., it is not a basic sequent). Since x and y are distinct free variables, then they can be "bound" to distinct terms, say s and t where neither s nor t occurs in A. A(s) and A(t) are clearly not syntactically identical in that they differ in the constants s and t.

Some consequences of the infon axiom system

An infon is considered to be *factual* if it is supported by some situation. Let F be some infon. Let $\boldsymbol{f}$ be $(F \wedge \overline{F})$. Since all situations are consistent (see Support condition 1, page 36), $\boldsymbol{f}$ can never be factual, regardless of the choice of infon F. Consider the compound infon $(A \Rightarrow \boldsymbol{f})$. By the definition of the support of confirmation of a conditional, $(A \Rightarrow \boldsymbol{f})$ is supported by a situation s if and only if there is no situation t of which s is a part such that $t \models A$ and $t \not\models \boldsymbol{f}$. Since $t \not\models \boldsymbol{f}$ holds for all t, this condition can be simplified to: there is no situation t of which s is a part such that $t \models A$. Suppose there exists a situation r such that $r \models A$. Since the "set" of situations is closed with respect to the union of situations, there exists a situation t which contains both *r and s*. By the persistence of infons, $r \models A$ implies $t \models A$. Thus, there exists a situation t of which s is a part such that $t \models A$. This implies that $s \not\models (A \Rightarrow \boldsymbol{f})$. Thus, if A is supported by *any* situation, no situation can support $(A \Rightarrow \boldsymbol{f})$. By contraposition, if any situation supports $(A \Rightarrow \boldsymbol{f})$, then no situation supports A. It is also easy to show that if no situation supports A, then any situation supports $(A \Rightarrow \boldsymbol{f})$. Thus, A is not factual if and only if $(A \Rightarrow \boldsymbol{f})$ is supported by all situations. The denial of the claim that an infon A is not factual, $-(A \Rightarrow \boldsymbol{f})$, is equivalent to confirmation of the claim that A is factual. It can also be shown that if $s \models A$ then $s \models -(A \Rightarrow \boldsymbol{f})$. This latter claim is what one might expect, that if a situation supports an infon, it also supports the claim that that infon is factual.

Gabbay notes that, in *nh*, $A \Leftrightarrow B$ does *not* imply $-A \Leftrightarrow -B$. It is also the case that $(A \Rightarrow B)$ does *not* imply $(-B \Rightarrow -A)$. Thus, one cannot use the contrapositive rule when reasoning in infon logic.

For classical logic it is sufficient to define a single connective via an appropriate axiom system, and all of the standard connectives can be defined in terms of that single connective. More commonly, axiom systems are used in classical logic which define only two connectives and all other connective are defined in terms of those two. The question naturally arises then of what the minimum set of connectives is for the infon logic. There is no axiom system which is logically equivalent to the infon axiom system which uses fewer connectives than that given in the propositional fragment of the infon axiom system (noting that '$\Leftrightarrow$' is a defined symbol in the infon axiom system).

This can be improved on for the *h* axiom system by introducing quantification over propositions, making the logic second order. Gabbay does this to create a second order *propositional* logic, *2h*. He extends the propositional fragment of HPC, *h*, with quantification of propositions to create *C2h*. He also extends the propositional Kripke structure to define an interpretation for *C2h*.
The axiom system for *C2h* is that of *h* extended as follows[39]:

$C2h_0 = h_0 \cup$

$\{(\forall x)A(x) \Rightarrow A(y),$

$A(y) \Rightarrow \exists x\, A(x),$

$\forall\, x\, (B \vee A(x)) \Rightarrow (B \vee \forall x A(x))$ [x not free in B],

$\exists x\, (x \Leftrightarrow A)$ [A any formula, x not free in A]$\}$

$C2h_1 = h_1 \cup$

$\{(A(x) \Rightarrow B\, /\, \exists x\, A(x) \Rightarrow B),$

$(B \Rightarrow A(x)\, /\, \mathrm{B} \Rightarrow \forall x A(x))\}$, x not free in B.

$C2h_2 = h_2$.

x and y are *propositional* variables in the above axioms and rules.

In this logic, '$\Rightarrow$' and '$\forall$' are the only connectives one needs to define the connectives of h [40]:

$$A \wedge B =_{\mathrm{df}} \forall\, x\, ((A \Rightarrow (B \Rightarrow x)) \Rightarrow x)$$

$$A \vee B =_{\mathrm{df}} \forall\, x\, ((A \Rightarrow x) \wedge (B \Rightarrow x) \Rightarrow x)$$

$$\exists x\, A(x) =_{\mathrm{df}} \forall y\, (\forall x(A(x) \Rightarrow y) \Rightarrow y)$$

$$f =_{\mathrm{df}} (\forall x)\, x$$

[39] p. 159 in [Gabbay 1981].
[40] p. 169 in [Gabbay 1981].

It appears that the above equivalences are not applicable to infon logic. In the case of conjunction, for instance, there is a semantic mismatch between the support of a conjunction and the support of a conditional. The support of a *conjunction* by some situation s is completely determined by that situation, "locally" as it were. The determination of support of a *conditional* by some situation s is dependent on the situations of which s is a part, a more "global" concern. Thus, if one has an infon A which is not supported by s but is supported by t ($t \geq_S s$), and B which *is* supported by s, then '$A \wedge B$' is not supported by s but '$\forall x\ ((A \Rightarrow (B \Rightarrow x)) \Rightarrow x)$' *is* supported by s (this latter claim is a little awkward to demonstrate, but follows from the support condition definitions).[41]

[41] This idea of quantifying over propositions is more interesting in the intuitionistic logics than in classical logic. In classical logic a proposition is either true or false (in a given interpretation). Thus, '$\forall x\ A(x)$', for x a propositional variable, is classically equivalent in truth value to 'A(true) $\wedge$ A(false)'. Similarly, '$\exists x\ A(x)$' is classically equivalent to 'A(true) $\vee$ A(false)'. In the infon logic, the dual notions of interest are "supported" versus "unsupported" (instead of "true" and "false"). There is a similar analysis as that for classical logics, but it is more complex due to the more complex definition of the semantic interpretation.

Langholm's Partial Model Theory

Tore Langholm published a work on the theory of partial models for logic that was inspired by situation theory titled *Partiality, Truth and Persistence* [Langholm 1988]. Various of the results of Langholm are comparable with those developed for the NH axiom system and the support conditions.[42] In the following, the logic semantics developed by Langholm is referred to as PM (for "Partial Model"). The major point about PM in the context of this thesis is that PM does not have a persistent definition of the conditional (NH, of course, does).

Truth conditions for a partial propositional logic

The basic definition is for the *strong Kleene* truth relation between wffs and models:

$v \models S^+$ iff $S \in v^+$

$v \models S^-$ iff $S \in v^-$

$v \models T^+$

$v \not\models T^-$

$v \models \neg\varphi^+$ iff $v \models \varphi^-$

$v \models (\varphi \vee \psi)^+$ iff $v \models \varphi^+$ or $v \models \psi^+$

$v \models (\varphi \vee \psi)^-$ iff $v \models \varphi^-$ and $v \models \psi^-$

The model v is defined by a triplet, $v = (\rho_v, v^+, v^-)$, where ρ_v is a set of atomic sentences (without polarity), $\rho_v \supseteq v^+ \cup v^-$ and $v^+ \cap v^- = \emptyset$. If $\rho_v = v^+ \cup v^-$, then v is a complete model. He defines "*v informationally extends u*", written $u \ll v$, if $\rho_u = \rho_v$, $v^+ \supseteq u^+$ and $v^- \supseteq u^-$. The model v can be considered to represent a situation *s*, where $v^+ \cup v^-$ is all of the basic infons which *s* supports. The similarity type ρ_v is just all of the basic infons (without polarity) which *any* situation supports. Thus, all situations have the same similarity type. The "part of" relation between situations is modeled by the "informationally extends" relation.

[42] Langholm makes no reference to the intuitionistic tradition in logic, notably Brouwer or Heyting, or the work of Gabbay. He includes a work by Melvin Fitting on intuitionism in his bibliography, but I was unable to locate any actual reference in [Langholm 1988] to Fitting's work.

Langholm introduces *exclusion negation*, written '~', with the definition:

$v \models \sim\varphi^+$ iff $v \not\models \varphi^+$

$v \models \sim\varphi^-$ iff $v \models \varphi^+$

This operator is also known as *external negation*.

The defined operators are:

$\bot =_{df} -T$

$\varphi \wedge \psi =_{df} -(-\varphi \vee -\psi)$

$\varphi \supset \psi =_{df} (\sim\varphi \vee \psi)$

$\varphi \equiv \psi =_{df} (\varphi \supset \psi) \wedge (\psi \supset \varphi)$

$\varphi \leftrightarrow \psi =_{df} (\varphi \equiv \psi) \wedge (-\varphi \equiv -\psi)$

'$\varphi \equiv \psi$' holds when φ and ψ are true in exactly the same models. This is read as "j and y are *positively equivalent*". Similarly, '$-\varphi \equiv -\psi$' holds when φ and ψ are false in exactly the same models. This is read as "φ and ψ are *negatively equivalent*". A second kind of disjunction, *weak disjunction*, is defined in terms of the primitive negation and (strong) disjunction:

$(\varphi \nabla \psi) =_{df} ((\varphi \vee \psi) \wedge (\varphi \vee -\varphi) \wedge (\psi \vee -\psi))$

Comparison of PM and NH

There is a great similarity between Langholm's strong Kleene structure and the strong propositional Kripke interpretation. They are not the same, though. The strong negation, conjunction and (strong) disjunction are the same for NH and Langholm's "partial model" logic (PM). There is no counterpart in NH for the exclusion negation of PM. Beyond this, there is nothing in NH which treats the positive and negative polarities of an infon asymmetrically. Exclusion negation does treat them asymmetrically. Thus, to define exclusion negation for NH requires modifying the strong propositional Kripke interpretation (extending it). However, exclusion negation is *not* persistent. This can be shown as follows: Let $\varphi \in \rho_u$, $\varphi \notin u^+$, and $\varphi \notin u^-$. This implies that $u \not\models \varphi^+$, and thus $u \models \sim\varphi^+$. Let $v^+ = u^+ \cup \{\varphi\}$, v- = u-, and $\rho_u = \rho_v$. This implies that $u << v$. However, it also implies that $v \models \varphi^+$. Thus, $v \not\models \sim\varphi^+$. Therefore,

‘~’ is not persistent.

The definitions of ‘⊃’ in PM and ‘⇒’ in NH are not directly comparable. They both support the deduction theorem, but their semantics are quite different. Most importantly, ‘⊃’ is not persistent. This can be shown in a manner analogous to the non-persistence of ‘~’. Similarly, ≌’ is not directly comparable with its NH analog ‘⇔’, and ‘≡’ and ‘↔’ are not persistent.[43]

Langholm does introduce another truth value relation which involves quantification over models in a fashion reminiscent of the Kripke interpretation. This is the *supervaluation* truth relation, written ‘$|=_{SV}$’[44]. It is defined by: $v \models_{SV} \varphi$ iff ($v' \models \varphi$ for all completions v' of v).[45] Langholm shows that : $v \models_{SV} \varphi$ iff $v \models \varphi$, in the propositional case. He mentions that this breaks down in the predicate logic case.

Langholm doesn’t retain persistence for the fundamental notion of the conditional, but it’s not clear what he gets for yielding up this property. This difference is sufficiently crucial as to make further comparisons between PM and NH unrewarding.

[43] These non-persistence results and much more are discussed starting on p. 26 in [Langholm 1988].
[44] Langholm writes it with a small box subscript.
[45] p. 36 in [Langholm 1988].

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Appendix 1: Infon Logic Theorem Proofs

This appendix contains the proof of the theorems of Chapter 3, the presentation of infon logic.

Theorem 1: All conditional-free infons are persistent.

Choose situations s, t such that $s \leq_S t$.
The theorem requires that for all conditional-free infons σ, if $s \models \sigma$ then $t \models \sigma$.

Proof:
The proof is by induction on the structure of infons.

Basis: σ is a basic infon. σ is a basic infon (positive or negative) implies $t \models \sigma$. [By definition of supports.]
QED, basis case.

Induction Hypothesis: If A, $\overline{A}$, B, and $\overline{B}$ are persistent infons, then $A \wedge B$, $-(A \wedge B)$, $A \vee B$, and $-(A \vee B)$ are persistent.

1) $s \leq_S t$. [By the antecedent of the theorem.]
2) $s \models A \wedge B$ iff $s \models A$ and $s \models B$. [By definition of confirmation of conjunction.]
3) $s \models A$ implies $t \models A$. [antecedent of induction hypothesis.]
4) $s \models B$ implies $t \models B$. [antecedent of induction hypothesis.]
5) $s \models A \wedge B$ implies $t \models A$ and $t \models B$. [By steps 2,3,4 and transitivity of implies.]
6) $t \models A \wedge B$ iff $t \models A$ and $t \models B$. [By definition of confirmation of conjunction.]
7) $s \models A \wedge B$ implies $t \models A \wedge B$. [By steps 5 and 6 and transitivity of implies.]
8) $A \wedge B$ is persistent. [By steps 1 and 7 and definition of persistence.]
9) $s \models -(A \wedge B)$ iff $s \models \overline{A}$ or $s \models \overline{B}$. [By definition of denial of conjunction.]
10) $s \models \overline{A}$ implies $t \models \overline{A}$. [antecedent of induction hypothesis.]
11) $s \models \overline{B}$ implies $t \models \overline{B}$. [antecedent of induction hypothesis.]
12) $s \models -(A \wedge B)$ implies $t \models \overline{A}$ or $t \models \overline{B}$. [By steps 9,10,11 and transitivity of implies.]
13) $t \models -(A \wedge B)$ iff $t \models \overline{A}$ or $t \models \overline{B}$. [By definition of denial of conjunction.]
14) $s \models -(A \wedge B)$ implies $t \models -(A \wedge B)$. [By steps 12 and 13 and transitivity of implies.]
15) $-(A \wedge B)$ is persistent. [By steps 1 and 14 and definition of persistence.]
16) $s \models A \vee B$ iff $s \models A$ or $s \models B$. [By definition of confirmation of disjunction.]
17) $s \models A \vee B$ implies $t \models A$ or $t \models B$. [By steps 2,3,16 and transitivity of implies.]
18) $t \models A \vee B$ iff $t \models A$ or $t \models B$. [By definition of confirmation of disjunction.]
19) $s \models A \vee B$ implies $t \models A \vee B$. [By steps 17 and 18 and transitivity of implies.]
20) $A \vee B$ is persistent. [By steps 1 and 19 and definition of persistence.]

21) $s \models –(A \vee B)$ iff $s \models \overline{A}$ and $s \models \overline{B}$. [By definition of denial of disjunction.]
22) $s \models –(A \vee B)$ implies $t \models \overline{A}$ and $t \models \overline{B}$. [By steps 9,10,21 and transitivity of implies.]
23) $t \models –(A \vee B)$ iff $t \models \overline{A}$ and $t \models \overline{B}$. [By definition of denial of disjunction.]
24) $s \models –(A \vee B)$ implies $t \models –(A \vee B)$. [By steps 22 and 23 and transitivity of implies.]
25) $–(A \vee B)$ is persistent. [By steps 1 and 24 and definition of persistence.]
26) $A \wedge B$, $–(A \wedge B)$, $A \vee B$, and $–(A \vee B)$ are persistent. [By steps 8, 15, 20, and 25.]

QED, the induction hypothesis.

From the basis case and the induction hypothesis, it follows that all conditional-free infons are persistent.

QED, Theorem 1.

Theorem 2: For all wffs A and situations t, if ($\alpha(t, A) = 1$ iff $[A]_t = 1$) and ($\alpha(t, A) = -1$ iff $[A]_t = -1$), then ($\alpha(t, A) = [A]_t$).

Proof:
1) $\alpha(t, A) = 0$ iff $\alpha(t, A) \neq 1$ and $\alpha(t, A) \neq -1$. [By definition, α has a range of {-1, 0, 1}, and α is total for all wffs A.]
2) $[A]_t = 0$ iff $[A]_t \neq 1$ and $[A]_t \neq -1$. [By definition, [] has a range of {-1, 0, 1}, and a is total for all wffs A.]
3) $\alpha(t, A) \neq 1$ iff $[A]_t \neq 1$. [By hypothesis of theorem and classical logic negational equivalence, (A iff B) iff (~A iff ~B).]
4) $\alpha(t, A) \neq -1$ iff $[A]_t \neq -1$. [By hypothesis of theorem and classical logic negational equivalence, (A iff B) iff (~A iff ~B).]
5) $\alpha(t, A) = 0$ iff $[A]_t \neq 1$ and $[A]_t \neq -1$. [By steps 1, 3, and 4, and (indirectly) transitivity of conditional.]
6) $\alpha(t, A) = 0$ iff $[A]_t = 0$. [By steps 2 and 5, and transitivity of equivalence.]
7) $\alpha(t, A) = [A]_t$. [By step 6, the theorem hypothesis, the totality of both α and [] over the same domain (the set of all wffs), and the definitions of both α and [] to have {-1, 0, 1} as their ranges.]

QED, Theorem 2.

Theorem 3: For all conditional-free A, $[A]_t = \alpha(\mathrm{t}, A)$.

Proof:
The proof of this is inductive on the connective structure of A, as follows:

Basis: A is atomic (no connectives). This implies that A is a basic infon.$[A]_t = \alpha(\text{t}, A)$ holds by definition.

Induction: Hypothesis: If $[A]_t = \alpha(\text{t}, A)$ and $[B]_t = \alpha(\text{t}, B)$, then $[A \wedge B]_t = \alpha(\text{t}, A \wedge B)$ and $[A \vee B]_t = \alpha(\text{t}, A \vee B)$ and $[–A]_t = \alpha(t, –A)$

1) $[A]_t = \alpha(\text{t}, A)$ and $[B]_t = \alpha(\text{t}, B)$. [Hypothesis, antecedent.]
2) $[A \wedge B]_t = 1$ iff $[A]_t = 1$ and $[B]_t = 1$. [By definition point 2 and definition of ‘min’.]
3) $\alpha(t, A \wedge B) = 1$ iff $t \models A \wedge B$. [By definition of α]
4) $t \models A \wedge B$ iff $t \models A$ and $t \models B$. [By Support condition 6, page 41, ‘Confirmation of Conjunction’.]
5) $\alpha(t, A \wedge B) = 1$ iff $t \models A$ and $t \models B$. [By steps 3 and 4, and transitivity of ‘iff’.]
6) $\alpha(t, A) = 1$ iff $t \models A$. [By definition of α.]
7) $\alpha(t, B) = 1$ iff $t \models B$. [By definition of α.]
8) $\alpha(t, A \wedge B) = 1$ iff $\alpha(t, A) = 1$ and $\alpha(t, B) = 1$. [By steps 5, 6, and 7, and by two applications of the theorem “((P iff (Q and R)) and (Q iff S)) implies (P iff (S and R))” (which follows from the transitivity of “implies”).]
9) $\alpha(t, A \wedge B) = 1$ iff $[A]_t = 1$ and $[B]_t = 1$. [By substituting equalities of step 1 into step 8.]
10) $\alpha(t, A \wedge B) = 1$ iff $[A \wedge B]_t = 1$. [By steps 2 and 9 and transitivity of ‘iff’.]
11) $[A \wedge B]_t = -1$ iff $[A]_t = -1$ or $[B]_t = -1$. [By definition point 2 and definition of ‘min’.]
12) $\alpha(t, A \wedge B) = -1$ iff $t \models –(A \wedge B)$. [By definition of α]
13) $t \models –(A \wedge B)$ iff $t \models \overline{A}$ or $t \models \overline{B}$. [By Support condition 7, page 42, ‘Denial of Conjunction’.]
14) $\alpha(t, A \wedge B) = -1$ iff $t \models \overline{A}$ or $t \models \overline{B}$. [By steps 12 and 13, and transitivity of ‘iff’.]
15) $\alpha(t, A) = -1$ iff $t \models \overline{A}$. [By definition of α.]
16) $\alpha(t, B) = -1$ iff $t \models \overline{B}$. [By definition of α.]
17) $\alpha(t, A \wedge B) = -1$ iff $\alpha(t, A) = -1$ or $\alpha(t, B) = -1$. [By steps 14, 15, and 16, and by two applications of the classical theorem “((P iff (Q or R)) and (Q iff S)) implies (P iff (S or R))” (which follows from the transitivity of “implies”).]
18) $\alpha(t, A \wedge B) = -1$ iff $[A]_t = -1$ or $[B]_t = -1$. [By substituting equalities of step 1 into step 30.]
19) $\alpha(t, A \wedge B) = -1$ iff $[A \wedge B]_t = -1$. [By steps 11 and 18 and transitivity of ‘iff’.]
20) $\alpha(t, A \wedge B) = [A \wedge B]_t$. [By steps 10 and 19, and Theorem 2, page 43.]
21) $[A \vee B]_t = 1$ iff $[A]_t = 1$ or $[B]_t = 1$. [By definition point 2 and definition of ‘min’.]
22) $\alpha(t, A \vee B) = 1$ iff $t \models A \vee B$. [By definition of α]

23) $t \models A \vee B$ iff $t \models A$ or $t \models B$. [By Support condition 6, page 41, 'Confirmation of Disjunction'.]

24) $\alpha(t, A \vee B) = 1$ iff $t \models A$ or $t \models B$. [By steps 22 and 23, and transitivity of 'iff'.]

25) $\alpha(t, A \vee B) = 1$ iff $\alpha(t, A) = 1$ or $\alpha(t, B) = 1$. [By steps 24, 6, and 7, and by two applications of the classical theorem "((P iff (Q and R)) and (Q iff S)) implies (P iff (S and R))" (which follows from the transitivity of "implies").]

26) $\alpha(t, A \vee B) = 1$ iff $[A]_t = 1$ or $[B]_t = 1$. [By substituting equalities of step 1 into step 25.]

27) $\alpha(t, A \vee B) = 1$ iff $[A \vee B]_t = 1$. [By steps 21 and 25 and transitivity of 'iff'.]

28) $[A \vee B]_t = -1$ iff $[A]_t = -1$ and $[B]_t = -1$. [By definition point 2 and definition of 'max'.]

29) $\alpha(t, A \vee B) = -1$ iff $t \models -(A \vee B)$. [By definition of α]

30) $t \models -(A \vee B)$ iff $t \models \overline{A}$ and $t \models \overline{B}$. [By Support condition 9, page 42, 'Denial of Disjunction'.]

31) $\alpha(t, A \vee B) = -1$ iff $t \models \overline{A}$ and $t \models \overline{B}$. [By steps 29 and 30, and transitivity of 'iff'.]

32) $\alpha(t, A \vee B) = -1$ iff $\alpha(t, A) = -1$ or $\alpha(t, B) = -1$. [By steps 31, 15, and 16, and by two applications of the classical theorem "((P iff (Q and R)) and (Q iff S)) implies (P iff (S and R))" (which follows from the transitivity of "implies").]

33) $\alpha(t, A \vee B) = -1$ iff $[A]_t = -1$ and $[B]_t = -1$. [By substituting equalities of step 1 into step 32.]

34) $\alpha(t, A \vee B) = -1$ iff $[A \vee B]_t = -1$. [By steps 28 and 33 and transitivity of 'iff'.]

35) $\alpha(t, A \vee B) = [A \vee B]_t$. [By steps 27 and 34, and Theorem 2, page 43.]

36) $[-A]_t = -[A]_t$. [By definition.]

37) $-[A]_t = -1$ iff $[A]_t = 1$. [By step 36 and arithmetic,X = Y iff A*X = A*Y.]

38) $[A]_t = 1$ iff $t \models A$. [By step 1, definition of α, and transitivity of iff.]

39) $[-A]_t = 1$ iff $[A]_t = -1$. [By step 36 and arithmetic,X = Y iff A*X = A*Y.]

40) $[A]_t = -1$ iff $t \models -A$. [By step 1, definition of α, and transitivity of iff.]

41) $\alpha(t, -A) = 1$ iff $t \models -A$. [By definition of α.]

42) $\alpha(t, -A) = -1$ iff $t \models A$. [By definition of α.]

43) $\alpha(t, -A) = 1$ iff $[-A]_t = 1$. [By steps 39, 40, and 41, and transitivity of iff.]

44) $\alpha(t, -A) = -1$ iff $[-A]_t = -1$. [By steps 36, 37, 38, and 42, and transitivity of iff.]

45) $\alpha(t, -A) = [-A]_t$. [By 43 and 44 and Theorem 2, page 43.]

46) If $[A]_t = \alpha(\mathrm{t}, A)$ and $[B]_t = \alpha(\mathrm{t}, B)$, then $[A \wedge B]_t = \alpha(\mathrm{t}, A \wedge B)$ and $[A \vee B]_t = \alpha(\mathrm{t}, A \vee B)$ and $[-A]_t = \alpha(t, -A)$. [By 20, 35, and 45.]

QED, the induction hypothesis.

Thus, reasoning by induction from the basis statement and the induction hypothesis, $[A]_t = \alpha(\mathrm{t}, A)$ for all conditional free A.

QED, Theorem 3.

Theorem 4: Propositional infons are persistent.

Proof: The proof of this is built on the proof that conditional-free infons are persistent. This proof is also done via structural induction.

Basis: If A is a basic infon, then A is persistent. This holds by Support condition 3.

Induction Hypothesis: If σ and $\bar{\tau}$ are persistent, then $\sigma \Rightarrow \tau$ and $-(\sigma \Rightarrow \tau)$ are persistent.

1) $s \models \sigma \Rightarrow \tau$ implies for all t such that $s \leq_S t$, $t \models \sigma$ implies $t \models \tau$. [By the confirmation of conditional support condition.]
2) For all s' and t, if s' contains s and t contains s', then t contains s. [By the transitive property of the 'part of' relation.]
3) $s \models \sigma \Rightarrow \tau$ implies for all s' and t, if s' contains s and $s' \leq_S t$, $t \models \sigma$ implies $t \models \tau$. [By steps 1 and 2.]
4) For all s', if s' contains s, then $s \models \sigma \Rightarrow \tau$ implies for all t, if $s' \leq_S t$, $t \models \sigma$ implies $t \models \tau$. [By step 3.]
5) For all s', if s' contains s, then $s \models \sigma \Rightarrow \tau$ implies $s' \models \sigma \Rightarrow \tau$. [By step 4 and Support condition 10.]
6) $s \models -(\sigma \Rightarrow \tau)$ implies $s \models \sigma$ and $s \models \bar{\tau}$. [By Support condition 11.]
7) For all s', if s' contains s, then $s' \models \sigma$ and $s' \models \bar{\tau}$. [By induction hypothesis antecedent.]
8) $s' \models \sigma$ and $s' \models \bar{\tau}$ implies $s' \models -(\sigma \Rightarrow \tau)$. [By Support condition 11.]
9) For all s', if s' contains s, then $s \models -(\sigma \Rightarrow \tau)$ implies $s' \models -(\sigma \Rightarrow \tau)$. [By steps 6, 7, and 8.]
10) $\sigma \Rightarrow \tau$ and $-(\sigma \Rightarrow \tau)$ are persistent. [By steps 5 and 6.]

QED, induction hypothesis.

Since the induction hypothesis for the conditional-free infons is proved in Theorem 1, page 42, then by induction on the structure of infons via that hypothesis and the one above, all propositional infons are persistent.

QED, persistence of propositional infons, Theorem 4.

Theorem 5: If $\alpha(t, A) = [A]_t$ and $\alpha(t, B) = [B]_t$, then $\alpha(t, A \Rightarrow B) = [A \Rightarrow B]_t$.

1) $\alpha(s, A \Rightarrow B) = 1$ iff $s \models A \Rightarrow B$. [By definition of α.]
2) $s \models A \Rightarrow B$ iff for all t such that $s \leq_S t$, $t \models A$ implies $t \models B$. [By Support condition 10, page 48, *Confirmation of Conditional*.]
3) $\alpha(s, A \Rightarrow B) = 1$ iff for all t such that $s \leq_S t$, $t \models A$ implies $t \models B$.
4) $s \leq_S t$ iff $s\mathrm{R}t$. [By definition of the R relation.]

5) $\alpha(t, A) = 1$ iff $t \models A$. [By definition of α.]
6) $\alpha(t, B) = 1$ iff $t \models B$. [By definition of α.]
7) $\alpha(t, A) = [A]_t$. [By induction hypothesis.]
8) $\alpha(t, B) = [B]_t$. [By induction hypothesis.]
9) $[A]_t = 1$ iff $t \models A$. [By steps 5 and 7 and substitution of equality.]
10) $[B]_t = 1$ iff $t \models B$. [By steps 6 and 8 and substitution of equality.]
11) $\alpha(s, A \Rightarrow B) = 1$ iff for all t such that s R t, $[A]_t = 1$ implies $[B]_t = 1$. [By steps 3, 4, 9, and 10]
12) $[A \Rightarrow B]_t$=1 iff for all t such that s R t, $[A]_t = 1$ implies $[B]_t = 1$. [By definition.]
13) $[A \Rightarrow B]_t$=1 iff $\alpha(s, A \Rightarrow B) = 1$. [By steps 11 and 12 and transitivity of iff.]
14) $\alpha(s, A \Rightarrow B) = -1$ iff $s \models -(A \Rightarrow B)$. [By definition.]
15) $t \models -(A \Rightarrow B)$ iff $t \models \overline{A}$ and $t \models B$. [By definition.]
16) $\alpha(t, A) = -1$ iff $t \models \overline{A}$. [By definition.]
17) $\alpha(t, A \Rightarrow B) = -1$ iff $[A]_t = -1$ and $[B]_t = 1$. [By steps 6 and 16 into 15, and 15 into 14.]
18) $[A \Rightarrow B]_t = -1$ iff $[A]_t = -1$ and $[B]_t = 1$. [By definition.]
19) $[A \Rightarrow B]_t = -1$ iff $\alpha(t, A \Rightarrow B) = -1$. [By steps 17 and 18.]
20) $\alpha(t, A \Rightarrow B) = [A \Rightarrow B]_t$. [By steps 13 and 19, and Theorem 2, page 43.]

QED, Theorem 5; the '$\Rightarrow$' connective extension to Theorem 3, page 43.

Theorem 7: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of x to the constituents of s, $s \models (A(x) \Rightarrow \exists y\, A(y))[\boldsymbol{g}_s]$.

1) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (A(x) \Rightarrow \exists y\, A(y))[\boldsymbol{g}_s]$ iff for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to constituents of s, $s \models (A[\boldsymbol{g}_s] \Rightarrow \exists y\, A(y)[\boldsymbol{g}_s])$. [Property of anchors]
2) $s \models (A[\boldsymbol{g}_s] \Rightarrow \exists y\, A(y)[\boldsymbol{g}_s])$ iff for all t such that $s \leq_S t$, $t \models A[\boldsymbol{g}_s]$ implies $t \models \exists y\, A(y)[\boldsymbol{g}_s]$. [Support condition 10, page 48]
3) $t \models \exists y\, A(y)[\boldsymbol{g}_s]$ iff there exists some non-parametric *anchoring* $\boldsymbol{f}_t = \{\mathbf{y}/a\}$, $a \in$ *constituents*(t) such that $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [Support condition 14, page 50]
4) $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$ implies there exists some non-parametric *anchoring* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t) such that $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [From the tautology 'P implies P'.]
5) $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$ implies $t \models \exists y\, A(y)[\boldsymbol{g}_s]$. [By steps 3 and 4 and the transitivity of conditional.]
6) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (A(x) \Rightarrow \exists y\, A(y))[\boldsymbol{g}_s]$. [By steps 1, 2, and 5].

QED, axiom = ($A(x) \Rightarrow \exists y\, A(y)$).

Theorem 8: For all s and non-parametric $\boldsymbol{g}_s$, $s \models (\forall y\, A(y) \Rightarrow A(x))[\boldsymbol{g}_s]$.

1) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models \forall y\, A(y) \Rightarrow A(x)$ iff for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models \forall y\, A(y)[\boldsymbol{g}_s] \Rightarrow A[\boldsymbol{g}_s]$. [Property of anchors.]
2) $s \models \forall y\, A(y)[\boldsymbol{g}_s] \Rightarrow A[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, $t \models \forall y\, A(y)[\boldsymbol{g}_s]$ implies $t \models A[\boldsymbol{g}_s]$. [Support condition 10, page 48]
3) $t \models \forall y\, A(y)[\boldsymbol{g}_s]$ iff for all situations r and non-parametric *anchorings* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t), $t \leq_S r$ implies $r \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$.
4) (For all situations r and non-parametric *anchorings* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t), $t \leq_S r$ implies $r \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$) implies $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [Since $t \leq t$, if the antecedent is true, then $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$ is true. Thus, the implication holds.]
5) $t \models \forall y\, A(y)[\boldsymbol{g}_s]$ implies $t \models A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [By steps 3 and 4.]
6) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (\forall y\, A(y) \Rightarrow A(x))[\boldsymbol{g}_s]$. [By steps 1, 2, and 5.]

QED, axiom = ($\forall y\, A(y) \Rightarrow A(x)$).

Theorem 9: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, $s \models (-\exists x A(x) \Leftrightarrow \forall x\, -A(x))$.

1) $s \models (-\exists x A(x) \Rightarrow \forall x\, -A(x))[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, $t \models -\exists x A(x)[\boldsymbol{g}_s]$ implies $t \models \forall x\, -A(x)[\boldsymbol{g}_s]$. [Support condition 10, page 48 and property of anchors]
2) $t \models -\exists x A(x)[\boldsymbol{g}_s]$ iff for all situations r and non-parametric *anchorings* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(r), t part of r implies $r \models -A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [Support condition 15, page 50]
3) $t \models \forall x\, -A(x)[\boldsymbol{g}_s]$ iff for all situations r and non-parametric *anchorings* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(r), t part of r implies $r \models -A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [Support condition 12, page 50]
4) $t \models -\exists x A(x)[\boldsymbol{g}_s]$ iff $t \models \forall x\, -A(x)[\boldsymbol{g}_s]$. [By steps 2 and 3.]
5) $s \models (-\exists x A(x) \Rightarrow \forall x\, -A(x))[\boldsymbol{g}_s]$. [By steps 1 and 4.]
6) $s \models (\forall x\, -A(x) \Rightarrow -\exists x A(x))[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, $t \models \forall x\, -A(x)[\boldsymbol{g}_s]$ implies $t \models -\exists x A(x)[\boldsymbol{g}_s]$. [Support condition 10, page 48]
7) $s \models (\forall x\, -A(x) \Rightarrow -\exists x A(x))[\boldsymbol{g}_s]$. [By steps 6 and 4.]
8) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constit-

uents of s, s |= $(-\exists x A(x) \Leftrightarrow \forall x -A(x))[\boldsymbol{g}_s]$. [By steps 5 and 7.]

QED, axiom = $-\exists x A(x) \Leftrightarrow \forall x -A(x)$.

Theorem 10: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= $(-\forall x A(x) \Leftrightarrow \exists x -A(x))[\boldsymbol{g}_s]$.

1) s |= $(-\forall x A(x) \Rightarrow \exists x -A(x))[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, t |= $-\forall x A(x)[\boldsymbol{g}_s]$ implies t |=$\exists x -A(x)[\boldsymbol{g}_s]$. [Support condition 10, page 48, and property of anchors.]
2) t |= $-\forall x A(x)[\boldsymbol{g}_s]$ iff there exists a non-parametric *anchoring* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t), such that t |= $-A[\boldsymbol{g}_s][\boldsymbol{f}_t]$. [Support condition 13, page 50]
3) t |=$\exists x -A(x)[\boldsymbol{g}_s]$ iff there exists some non-parametric *anchoring* $\boldsymbol{f}_t = \{\mathbf{x}/a\}$, $a \in$ *constituents*(t) such that t |= $-A[\boldsymbol{g}_s][\boldsymbol{f}_t]$.
4) t |= $-\forall x A(x)[\boldsymbol{g}_s]$ iff t |=$\exists x -A(x)[\boldsymbol{g}_s]$. [By steps 2 and 3.]
5) s |= $(-\forall x A(x) \Rightarrow \exists x -A(x))[\boldsymbol{g}_s]$. [By steps 1 and 4.]
6) s |= $(\exists x -A(x) \Rightarrow -\forall x A(x))[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, t |= $\exists x -A(x)[\boldsymbol{g}_s]$ implies t |=$-\forall x A(x)[\boldsymbol{g}_s]$. [Support condition 10, page 48]
7) s |= $(\exists x -A(x) \Rightarrow -\forall x A(x))[\boldsymbol{g}_s]$. [By steps 4 and 6.]
8) For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= $(-\forall x A(x) \Leftrightarrow \exists x -A(x))[\boldsymbol{g}_s]$. [By steps 5 and 7.]

QED, axiom = $-\forall x A(x) \Leftrightarrow \exists x -A(x)$.

Theorem 11: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= $(A(x) \Rightarrow B)[\boldsymbol{g}_s]$ implies for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= $(\exists x A(x) \Rightarrow B)[\boldsymbol{g}_s]$.

0) Assume: $(A(x) \Rightarrow B)$ is a theorem. [By hypothesis.]
1) s |= $(A(x) \Rightarrow B)[\boldsymbol{g}_s]$ iff s |= $(A[\boldsymbol{g}_s] \Rightarrow B[\boldsymbol{g}_s])$. [Property of anchor.]
2) s |= $(A[\boldsymbol{g}_s] \Rightarrow B[\boldsymbol{g}_s])$ iff for all $\boldsymbol{g'}_s$ such that $\boldsymbol{g'}_s(y) = \boldsymbol{g}_s(y)$ for $y \neq x$, s |= $(A[\boldsymbol{g'}_s] \Rightarrow B[\boldsymbol{g}_s])$.
3) s |= $(A[\boldsymbol{g'}_s] \Rightarrow B[\boldsymbol{g}_s])$ iff for all t such that $s \leq_S t$, t |= $A[\boldsymbol{g'}_s]$ implies t |= $B[\boldsymbol{g}_s]$. [Support condition 10, page 48]
4) For all s and $\boldsymbol{g}_s$, there exists $\boldsymbol{g'}_s$ such that t |= $A[\boldsymbol{g'}_s]$ implies t |= $B[\boldsymbol{g}_s]$. [By steps 0, 1, 2, and 3.]
5) Suppose: there exists s_1, t_1, $\boldsymbol{g}_{s_1}$, $\boldsymbol{g}_{t_1}$, and $\boldsymbol{g'}_{t_1}$, $s_1 \leq_S t_1$, $\boldsymbol{g}_{s_1} = \boldsymbol{g}_{t_1}$, $t_1 \not\models \mathrm{B}[\boldsymbol{g}_{s_1}]$ and t_1 |= $\mathrm{A}[\boldsymbol{g'}_{t_1}]$.

6) ~(t_1 |= A[$\boldsymbol{g'}_{t_1}$] implies t_1 |= B[$\boldsymbol{g}_{s_1}$]). [By 5 and negation of implication.]

7) ~(t_1 |= A[$\boldsymbol{g'}_{t_1}$] implies t_1 |= B[$\boldsymbol{g}_{t_1}$]). [By 5 and 6 and substitution of equal terms (gs1 = gt1).]

8) Contradiction of step 4: ~(For all t such that $s \leq_S t$, t |= A[$\boldsymbol{g'}_s$] implies t |= B[$\boldsymbol{g}_s$]). [By 7, letting $s = t_1$, since $s \leq_S s$.]

9) ~(there exists s_1, t_1, $\boldsymbol{g}_{s_1}$, $\boldsymbol{g}_{t_1}$, and $\boldsymbol{g'}_{t_1}$, $s_1 \leq_S t_1$, $\boldsymbol{g}_{s_1} = \boldsymbol{g}_{t_1}$, t_1 |≠ B[$\boldsymbol{g}_{s_1}$] and t_1 |= A[$\boldsymbol{g'}_{t_1}$]). [Contradiction of step 5, by step 8's contradiction of 4.]

10) For all s_1, t_1, $\boldsymbol{g}_{s_1}$, $\boldsymbol{g}_{t_1}$, and $\boldsymbol{g'}_{t_1}$, ~($s_1 \leq_S t_1$ and $\boldsymbol{g}_{s_1} = \boldsymbol{g}_{t_1}$ and t_1 |≠ B[$\boldsymbol{g}_{s_1}$] and t_1 |= A[$\boldsymbol{g'}_{t_1}$]). [By step 9 and classical property of negation of universal quantification.]

11) For all s_1, t_1, $\boldsymbol{g}_{s_1}$, $\boldsymbol{g}_{t_1}$, and $\boldsymbol{g'}_{t_1}$, ($s_1 \leq_S t_1$ and $\boldsymbol{g}_{s_1} = \boldsymbol{g}_{t_1}$ and t_1 |= A[$\boldsymbol{g'}_{t_1}$]) implies t_1 |= B[$\boldsymbol{g}_{s_1}$]). [By step 10 and classical equivalence of 'implies' and 'or' (P -> Q iff (~P or Q)) and associativity of 'and'.]

12) For all s_1, t_1, $\boldsymbol{g}_{s_1}$, and $\boldsymbol{g}_{t_1}$, ($s_1 \leq_S t_1$ and $\boldsymbol{g}_{s_1} = \boldsymbol{g}_{t_1}$ and there exists $\boldsymbol{g'}_{t_1}$ such that t_1 |= A[$\boldsymbol{g'}_{t_1}$]) implies t_1 |≠ B[$\boldsymbol{g}_{s_1}$]). [By step 11 and classical equivalence of quantification of implication (for all x (P(x) -> Q) iff ((exists x P(x)) -> Q), x not free in Q).]

13) For all s and $\boldsymbol{g}_s$, s |= ($\exists x A(x) \Rightarrow B$)[$\boldsymbol{g}_s$] iff for all t such that $s \leq_S t$, t |= $\exists x A(x)$[$\boldsymbol{g}_s$] implies t |= B[$\boldsymbol{g}_s$]. [Support condition 10, page 48, and property of anchors.]

14) t |= $\exists x A(x)$[$\boldsymbol{g}_s$] iff there exists $\boldsymbol{g'}_t$ such that $\boldsymbol{g'}_s(y) = \boldsymbol{g}_s(y)$ for $y \neq x$, such that t |= A[$\boldsymbol{g'}_t$]. [By Support condition 14, page 50.]

15) For all s and $\boldsymbol{g}_s$, s |= ($\exists x A(x) \Rightarrow B$)[$\boldsymbol{g}_s$]. [By steps 12, 13 and 14, letting s_1 = s, t_1 = t, $\boldsymbol{g}_{s_1} = \boldsymbol{g}_s$, and $\boldsymbol{g'}_{t_1} = \boldsymbol{g'}_t$.]

16) ($\exists x A(x) \Rightarrow B$) is a theorem. [By step 15 and semantic definition of theorem.]

17) If ($A(x) \Rightarrow B$) is a theorem, then ($\exists x A(x) \Rightarrow B$) is a theorem. [By steps 0 and 16.]

QED, provability rule = ($A(x) \Rightarrow B$) / ($\exists x A(x) \Rightarrow B$).

Theorem 12: For all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= ($B \Rightarrow A(x)$)[$\boldsymbol{g}_s$] implies for all s and all non-parametric anchors $\boldsymbol{g}_s$ of parameters of A to the constituents of s, s |= ($B \Rightarrow \forall x A(x)$)[$\boldsymbol{g}_s$].

1) For all $\boldsymbol{g}_s$ in s, s |= ($B \Rightarrow A(x)$)[$\boldsymbol{g}_s$] iff s |= B[$\boldsymbol{g}_s$] $\Rightarrow$ A[$\boldsymbol{g}_s$]. [By anchor property.]

2) For all $\boldsymbol{g'}_s$ in s where $\boldsymbol{g'}_s(y) = \boldsymbol{g}_s(y)$ for $y \neq x$, s |= B[$\boldsymbol{g}_s$] $\Rightarrow$ A[$\boldsymbol{g}_s$] iff s |= B[$\boldsymbol{g}_s$]

$\Rightarrow A[\, \boldsymbol{g'}_s]$. [By given constraint that x is not free in B.]

3) $s \models B[\boldsymbol{g}_s] \Rightarrow A[\, \boldsymbol{g'}_s]$ iff for all t such that $s \leq_S t$, $t \models B[\boldsymbol{g}_s]$ implies $t \models A[\, \boldsymbol{g'}_s]$. [By Support condition 10, page 48.]

4) For all r such that $t \leq_S r$, $t \models B[\boldsymbol{g}_s]$ implies $r \models B[\boldsymbol{g}_s]$. [By persistence.]

5) $\boldsymbol{g}_s = \boldsymbol{g}_r$ an anchor in t and r. [Since $s \leq_S t$ and $s \leq_S r$, then the constituents of s are constituents of t and r. Thus an anchor in s is an anchor in t and r.]

6) For all s and $\boldsymbol{g}_s$ in s, $s \models (B \Rightarrow A(x))[\boldsymbol{g}_s]$ implies for all t and r such that $s \leq_S t \leq_S r$ and for all $\boldsymbol{g'}_r$ in r where $\boldsymbol{g'}_r(y) = \boldsymbol{g}_s(y)$ for $y \neq x$, $t \models B[\boldsymbol{g}_s]$ implies $r \models A[\, \boldsymbol{g'}_r]$. [By steps 1, 2,4, and 5.]

7) $s \models (B \Rightarrow \forall x A(x))[\boldsymbol{g}_s]$ iff $s \models B[\boldsymbol{g}_s] \Rightarrow \forall x A(x)[\boldsymbol{g}_s]$. [Property of anchors.]

8) $s \models B[\boldsymbol{g}_s] \Rightarrow \forall x A(x)[\boldsymbol{g}_s]$ iff for all t such that $s \leq_S t$, $t \models B[\boldsymbol{g}_s]$ implies $t \models \forall x A(x)[\boldsymbol{g}_s]$. [Support condition 10, page 48]

9) $t \models \forall x A(x)[\boldsymbol{g}_s]$ iff for all situations r and non-parametric *anchorings* $\boldsymbol{g'}_r$, where $\boldsymbol{g'}_r(y) = \boldsymbol{g}_s(y)$, for $y \neq x$, $\boldsymbol{g'}_r(x) \in$ *constituents*(r), $t \leq_S r$ implies $r \models A[\boldsymbol{g'}_r]$. [Support condition 12, page 50]

10) For all s and $\boldsymbol{g}_s$ in s, $s \models B[\boldsymbol{g}_s] \Rightarrow A[\, \boldsymbol{g}_s]$ implies for all s and $\boldsymbol{g}_s$ in s, $s \models B[\boldsymbol{g}_s] \Rightarrow \forall x A(x)[\boldsymbol{g}_s]$. [By steps 2, 6 and 9.]

11) For all s and $\boldsymbol{g}_s$ in s, $s \models (B \Rightarrow A(x))[\, \boldsymbol{g}_s]$ implies for all s and $\boldsymbol{g}_s$ in s, $s \models (B \Rightarrow \forall x A(x))[\boldsymbol{g}_s]$. [By steps 1, 7, and 10.]

QED, provability rule = ($B \Rightarrow A(x))/(B \Rightarrow \forall x A(x))$.